WORKING PAPER
## ALFRED P. SLOAN SCHOOL OF MANAGEMENT

A THEORETICAL AND EMPIRICAL INVESTIGATION
OF THE DUAL PURPOSE FUNDS:
AN APPLICATION OF CONTINGENT-CLAIMS ANALYSIS

Jonathan E. Ingersoll, Jr.[*]

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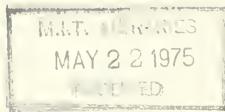
A Theoretical and Empirical Investigation

of the Dual Purpose Funds:

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Jonathan E. Ingersoll, Jr.

I   Introduction

Dual funds are a special type of closed end investment
company. Their purpose is to provide for investors with the
diverse objectives of long-term capital gains and present
income. This is accomplished through the formation of two
classes of shares: Capital shares which pay no dividends
and are redeemable at net asset value at maturity of the
fund and income shares which have the rights to all income
that the fund may earn, subject to a stated minimum
cumulative dividend and are to be redeemed at a set price at
the maturity of the fund.

Seven dual funds were formed in 1967, American Dual
Vest, Gemini, Income and Capital, Leverage Fund, Hemisphere,
Putnam Duo-Fund, and Scudder Duo-Vest. In addition there
are two dual funds which are also tax-free exchange funds;
these will not be considered here.

The capital shares of dual funds are entitled to no

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payments until maturity and then receive the entire fund
less repayment of the income shares. In this respect they
are analogous to warrants. In this paper the Black-Scholes
[1,2] warrant pricing equation is applied to the capital
shares in an attempt to explain their values and discounts.

Sections II and III develop the pricing equations for
the capital and income shares through option pricing
techniques. Section IV examines the characteristics of the
two claims that this model implies. The latter sections
apply the developed model to the seven dual purpose funds to
empirically evaluate the model.

II   The Capital Shares

To derive the dual fund pricing equations the following assumptions are required:

A 1   The capital markets are perfect in that: There are no taxes or transactions costs. Assets are perfectly divisible. Investors act as price takers. There is ulimited borrowing or lending at the risk free rate of interest. There are no restrictions on short sales.

A 2   Trading in assets takes place continuously in time.

A 3   The asset value, V, satisfies the stochastic differential equation:

$$dV = (\alpha V - C)dt + \sigma V dz$$

where C is the net cash flow paid out per unit time; α and σ^2 are the instantaneous expected rate of return and variance of return on the asset. σ is non-stochastic and at most a function of time. dz is a Gauss-Wiener process.

A 4   The term structure is flat and non-stochastic.[1]   r is the instantaneous riskless rate of interest.

Under these assumptions Merton |5| has shown that any contingent claim whose value can be written as a function of time and the asset value must satisfy the partial differential equation

$$(1) \qquad \frac{1}{2}\sigma^2 V^2 F_{VV} + (rV - C)F_V - rF + F_t + c = 0$$

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[1] This assumption is later modified to allow for stochastic interest rates. However, it had minimal effect in the pricing of the dual funds, and the strict assumption is kept here for expositional purposes.

where c is that portion of C payable to the contingent
claim, and subscripts denote partial derivatives.

If F has a contractual expiration or maturity date τ
periods in the future, then it may be written as $F(V,\tau)$ and
(1) becomes

$$(2) \qquad \tfrac{1}{2}\sigma^2 V^2 F_{VV} + (rV \cdot C)F_V - rF - F_\tau + c = 0$$

Equation (2), two boundary conditions, and one intial
condition are sufficient in principle to solve for F. In
general the boundary conditions are $F(0,\tau) = 0$ and $F(V,\tau) \leq V$
which are derived from the limited liability of F and of
the claim represented by V-F. The initial condition is
contained in the indenture agreement. The initial condition
and the functional form of c completely determine the
value of any contingent claim with the above simple boundary
conditions.

Solving (2) analytically is difficult with no further
assumptions. The following simplifying assumptions are made
here:

A 5 σ^2 is a constant.

A 6 Dividends are paid continuously to the income shares
and are proportional to V, $D = \delta V$. -2-

-2- Some of the funds have requirements that the fund's
assets be invested only in securities that currently earn at
least some minimum fixed proportio1 in dividends or
interest. If the managers, the large majority of which are
elected by the capital share owners, act to maximize the
capital shares' market value subject to this constraint,
they will insure that it is just met. The empirical content
of this assumption is examined in section V.

A 7   Management fees are paid continuously and are
      proportional to V, $M = \mu V$. -3-

For $f(V, \tau)$ the value of the capital shares of the dual
fund, $C = (\delta + \mu)V = \gamma V$ and $c = 0$. Then:

$$(3) \qquad \tfrac{1}{2}\sigma^2 V^2 f_{VV} + (r - \gamma)V f_V - rf - f_\tau = 0$$

subject to

$$(4) \qquad f(0, \tau) = 0 \;, \quad f(v, \tau) \leq V \;, \quad f(V, 0) = Max(0, V - E)$$

where E = the redemption price of the income share.   To
solve (3) we make the substitutions $X = V \cdot e^{-\gamma \tau}$, $g(X, \tau) = $
$f(V, \tau)$.   Then $f_V = g_X e^{-\gamma \tau}$, $f_{VV} = g_{XX} e^{-2\gamma \tau}$, $f_\tau = g_\tau - g_X \gamma e^{-\gamma \tau}$.

Substituting these values into (3) we derive

$$(5) \qquad \tfrac{1}{2}\sigma^2 X^2 g_{XX} + rX g_X - rg - g_\tau = 0$$

This equation is isomorphic to the European call option
equation on a stock paying no dividends.   Thus the solution
is

$$(6) \qquad f(V, \tau; E, r, \sigma^2, \gamma) = W(e^{-\gamma \tau}V, \tau; E, r, \sigma^2)$$

where W is the option solution in $|4|$.

As demonstrated therein, the option function is

-3- μ is generally one-half per cent per year. Some of
the funds have provisions by which this fraction can be
reduced; however, they have rarely if ever become effective.

homogeneous of degree one in V and $Ee^{-r\tau}$ hence f may be written

$$f(v,\tau) = Ee^{-r\tau}\left[z\,\phi(x_1) - \phi(x_2)\right]$$

where

$$z \equiv \frac{V}{E}\,e^{(r-\delta)\tau}$$

(7)

$$x_1 \equiv (\log z + \tfrac{1}{2}\sigma^2\tau)/\sigma\sqrt{\tau}$$

$$x_2 \equiv x_1 - \sigma\sqrt{\tau}$$

$$\phi(x) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp(-s^2/2)\,ds \; , \; \text{the cumulative normal distribution}$$

The behavior of f is shown in figures 1 and 2. Two things should be noted. The capital share is always more valuable than $\text{Max}(0, e^{-\delta\tau}V - Ee^{-r\tau})$. This is similar to the dominance constraint on European warrants, $f > \text{Max}(0, s - Ee^{-r\tau})$ Nevertheless, the capital shares can sell for less than their net asset value, defined by $V-E$ and indicated by the dashed line in figure 1, if the asset value is sufficiently large. We note that $f(V,) \sim Ve^{-\delta\tau} - Ee^{-r\tau}$ for large V; hence, the capital shares can sell at a discount if V is sufficiently larger than $E(1-\tilde{e}^{\tau})/(1-\tilde{e}^{\delta\tau})$. Finally we note that as the maturity date approaches, the solid line will coalesce with the dashed line and ceteris paribus the capital shares will sell at a smaller discount or at a premium.

The comparative statics for f are:

a) $f_v = e^{-\delta\tau}\,\phi(x_1) \geq 0$

Figure 1

1) $V > E$, $r > \gamma \frac{V}{E}$

2) $V > E$, $\frac{V}{E}\gamma > r > \sqrt{\frac{V}{E}}\gamma$

3) $V > E$, $\sqrt{\frac{V}{E}}\gamma > r$

4) $V < E$
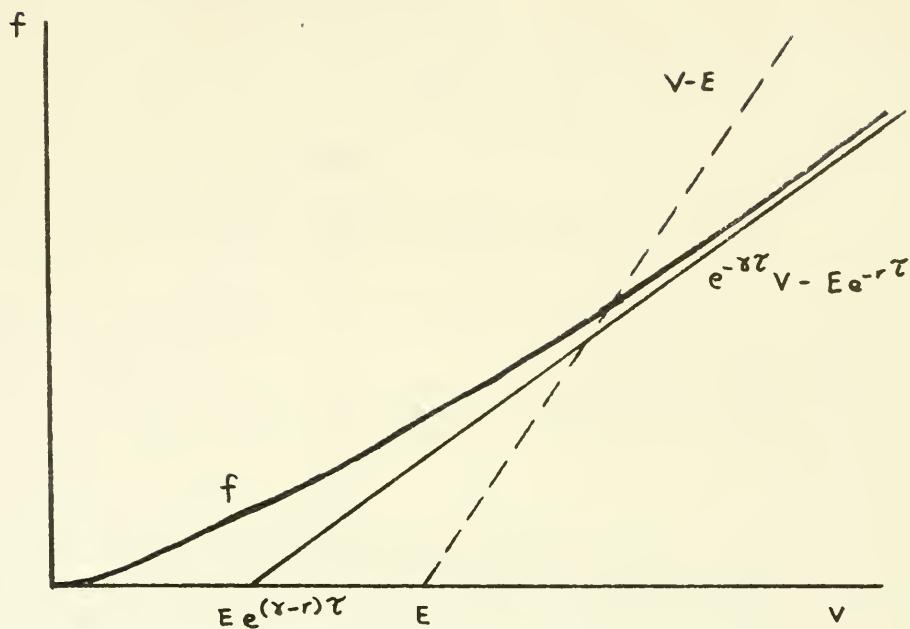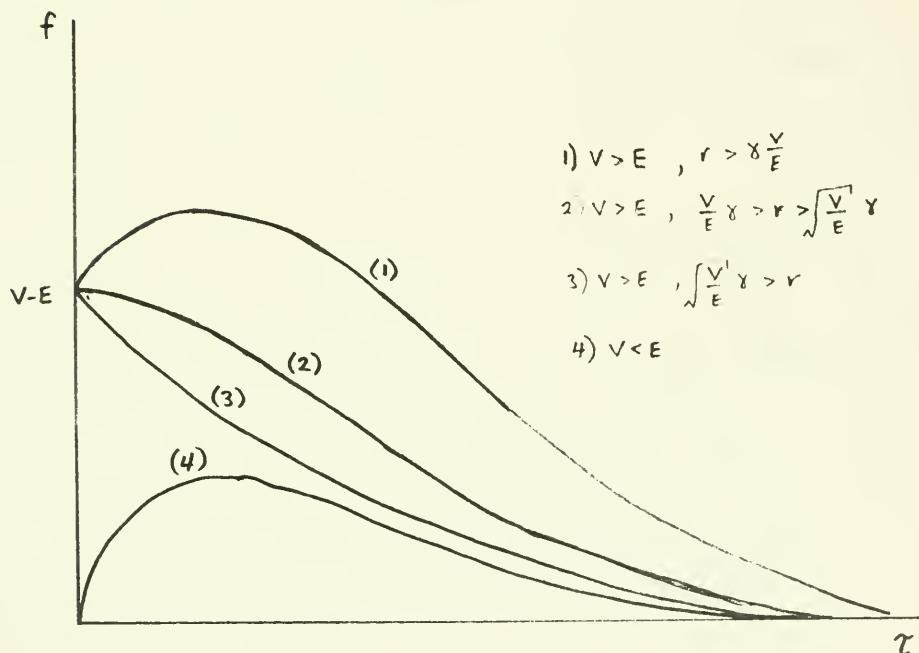
Figure 2

b) $f_{VV} = \dfrac{e^{-\gamma\tau}}{V\sigma\sqrt{\tau}}\, \Phi'(x_1) \geq 0$

c) $f_E = -e^{-r\tau}\, \Phi(x_2) \leq 0$

(8)

d) $f_r = -\tau E f_E \geq 0$

e) $f_{\sigma^2} = \dfrac{E e^{-r\tau}\sqrt{\tau}}{2\sigma}\, \Phi'(x_1) \geq 0$

f) $f_\gamma = -\tau V f_V \leq 0$

g) $f_\tau = \dfrac{V^2}{2}\sigma^2 f_{VV} + (r-\gamma)V f_V - rf \gtrless 0$

(Note: In all cases the inequalities will hold
strictly with the possible exceptions occuring
only at $V = 0$ or $\tau = 0$.)

(8a) through (8e) are the expected results from option
theory. f_γ has the sign one would expect, namely the value
of the capital shares is a decreasing function of the
dividends and fees paid.

Examining f_τ we find that it can be either positive
or negative. For small V the first term dominates and f_τ
is positive. This result is reasonable since given any
asset return dynamics a sufficiently small, though positive,
value could be chosen for V to virtually guarantee
default. In this case the capital share holders would
prefer a longer maturity to increase the probability that
the asset would grow enough in value to redeem the income
shares and still have a residual for the capital shares. On
the other hand, for large V, $f_\tau \sim -\gamma e^{-\gamma\tau} < 0$. Again this

is a sensible result since the "loan" represented by the income shares is virtually riskless in terms of default while the compensation provided by the proportional dividends from now until maturity will be in excess on the riskless rate.

III   The Income Shares

Under the same assumptions as in section II the income shares can be priced through the differential equation

(9) $\frac{1}{2}\sigma^2 V^2 F_{vv} + (r-\gamma)VF_v - rF - F_\tau + \delta V = 0$

subject to $F(0,\tau) = 0$, $F(v,\tau) \le V$, $F(v,o) = Min(V,E)$

Making the substitutions $X = Ve^{-\gamma\tau}$ and $G = X - F$, equation (9) becomes

(10) $\frac{1}{2}\sigma^2 X^2 G_{xx} + rXG_x - rG - G_\tau = -\delta X e^{\gamma\tau}$

subject to $G(0,\tau) = 0$, $G(x,\tau) \le X$, $G(x,o) = Max(0, x-E)$

Equation (10) is again the European option equation hence the two homogeneous solutions are $G = W(X,\tau)$ and $G = X$.   The particular solution is $-\frac{\delta}{\gamma}X e^{\gamma\tau}$.   Using the initial condition to determine the appropriate constants and solving back for F we derive

(11) $F(v,\tau) = V[e^{-\gamma\tau}(1-\frac{\delta}{\gamma}) + \frac{\delta}{\gamma}] - f(v,\tau)$

From (11) it is clear that the market value of the dual fund considered as a whole, $\mathcal{F} = F + f$, is linear in V.-4-

(12) $\mathcal{F}(v,\tau) = V[e^{-\gamma\tau}(1-\frac{\delta}{\gamma}) + \frac{\delta}{\gamma}] = A(\tau) V$

-4- $\mathcal{F}$ would be the market value of a closed-end fund of

The value of the future management fees is V-$\mathcal{F}$ or $\frac{\mu}{\gamma}(1-e^{-\gamma\tau})V$. Similarly the value of any pure dividend claim would be $\omega(1-e^{-\gamma\tau})V$ where ω is the fraction that the dividends are of the total cash flow. The value of the asset net of all cash payments is thus $e^{-\gamma\tau}V$ or what we have called X.

We can clarify the pricing of the income shares if we imagine them to be made up of two separable claims, one denoted h on the dividend stream and the other denoted H on the final redemption payment of E. From above, the pure dividend claim has the value

$$(13) \qquad h(v,\tau) = V\frac{\delta}{\gamma}(1-e^{-\gamma\tau}) = B(\tau)V$$

The redemption claim is a discount bond on the asset net of cash payments and from $|5|$ we can write

$$(14) \qquad H(v,\tau) = X - w(x,\tau) = e^{-\gamma\tau}V - f(v,\tau)$$

$\mathcal{F}$ and F are plotted in figure 3. The income share is a monotonic increasing, concave function of V. For small V the discount bond claim is approximately the entire market value net of the cash payments. For large asset values the discount bond becomes essentially riskless and $H \sim Ee^{-r\tau}$.

A plot of $\mathcal{F}$ vs. τ would show it to be negatively sloped and concave throughout starting from V with a slope

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the usual type (ie. with only one class of shares) having the same asset, dividend, and fee characteristics. Since A is less than one we can see that all closed-end funds should sell at a discount from their asset values.
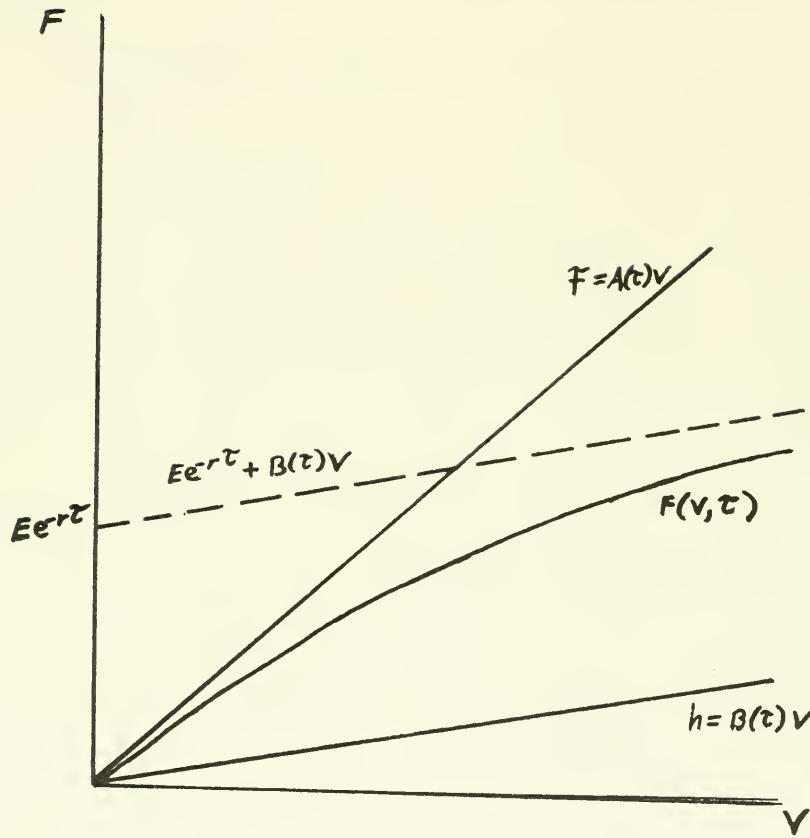
Figure 3

of $-\mu V$ at $\tau = 0$ and approaching $\frac{\delta}{\gamma} V$ as τ increases.
Since the management fees are generally one-half percent,
this plot would look virtually flat. Consequently the plot
of F would look essentially like those in figure 2 with
the vertical axis measured negatively away from V rather
than 0.

The comparative statics for $\tilde{F}$ are:

(15)
$$\tilde{F}_V = A > 0$$
$$\tilde{F}_\tau = -\mu V e^{-\gamma \tau} \leq 0$$
$$\tilde{F}_\delta = V \frac{\mu}{\gamma^2} \left[1 - e^{-\gamma \tau}(1 + \gamma \tau) \right] \geq 0$$
$$\tilde{F}_\mu = -V \frac{1}{\gamma} \left[\frac{\delta}{\gamma} (1 - e^{-\gamma \tau}) + \mu \tau e^{-\gamma \tau} \right] \leq 0$$
$$\tilde{F}_r = \tilde{F}_E = \tilde{F}_{\sigma^2} = 0$$

The comparative statics for F are then:

(16)
a) $F_V = \tilde{F}_V - f_V = \frac{\delta}{\gamma} (1 - e^{-\gamma \tau}) + e^{-\gamma \tau} \left[1 - \Phi(x_1) \right] > 0$

b) $F_{VV} = -f_{VV} \leq 0$

c) $F_E = -f_E \geq 0$

d) $F_r = -f_r \leq 0$

e) $F_{\sigma^2} = -f_{\sigma^2} \leq 0$

f) $F_\delta = \tilde{F}_\delta - f_\gamma \geq 0$

g) $F_\mu = \tilde{F}_\mu - f_\gamma = -V \frac{\delta}{\gamma^2} \left[1 - e^{-\gamma \tau}(1 + \gamma \tau) \right] - \left[1 - \Phi(x_1) \right] \tau V e^{-\gamma \tau} \leq 0$

h) $F_\tau = \tilde{F}_\tau - f_\tau = H_\tau + h_\tau \lessgtr 0$

i) $h_\tau = \delta V e^{-\gamma \tau} \geq 0$

j) $H_\tau = -\gamma V e^{-\gamma \tau} \left[1 - \Phi(x_1) \right] - r E e^{-r \tau} \Phi(x_2) - \frac{1}{2} \sigma^2 V^2 f_{VV} \leq 0$

The results in (16a) through (16e) are those expected
for any debt-like claim. From (16f) and (16g) the income
share is an increasing function of the dividends and a
decreasing function of the management fees. Again these are
expected. The ambiguous sign of F_τ is inherent in the
dual nature of the income share claim. When V is low, the
income share behaves like a discount bond since the
dividends are small; hence $F_\tau \sim H_\tau < 0$. If V is large
enough, then the value of the future dividends will dominate
in F, and the income share will behave like a pure dividend
claim; hence, $F \sim h_\tau > 0$ since the investor will receive the
dividend stream for a longer time if the maturity is
greater.

IV Fund Characteristics

In this section we shall examine some of the characteristics of dual purpose funds that are of concern to investors. One item that appears to be of particular concern is discounts. Discounts for these funds are of two types. One, common to all closed-end funds, is reflected in the difference between the market value of the fund and the value of the fund's holdings. (ie. the asset value of the fund) The other, more commonly referred to discount, is the capital share discount. It is defined as that percentage by which the market price of the capital share falls below its "net asset value." In the model presented in the previous sections these discounts would be

$$(17) \qquad \Delta \equiv 1 - \frac{\Upsilon}{V} \qquad\qquad \Delta_c \equiv 1 - \frac{f}{V - E} \qquad , V \geq E$$

The income shares do not have an associated discount discussed in the literature; however, a definition consistent with the above would be

$$(18) \qquad \Delta_I \equiv 1 - \frac{F}{E}$$

From section III we note that the fund will never sell at a premium and will sell at a discount except at the instant the fund matures since $\;0 \leq \Delta = (1 - e^{-\delta \tau})(1 - \delta/\gamma)$, from (12). As noted in the previous section, this property is not peculiar to the dual fund but is common to all

closed-end funds. Furthermore it is qualitatively independent of the proportionality assumptions previously placed on the dividends and fees. A closed-end fund will sell at a discount whenever (1) there are payments to which the owners of the funds are not entitled (eg. management fees) and (2) the fund shares are not redeemable at asset value. This follows from the realization that the future management fees must have a positive value regardless of their contractual structure or of dividend policy.-5-

The capital share discount appears to be of some importance both to investors and to the funds' managers. An American Dualvest Fund report stated for example, "...the capital shares continue to sell at a substantial discount from net asset value. Your management believes that this disparity is not justified by the performance of the fund." The belief that the capital shares should not sell at a discount appears quite widespread. It is perhaps associted with the true constraint that warrants sell for more than their "intrinsic value."-6- However, the latter is binding only from the arbitrage opportunity that would otherwise be present through the purchase and immediate exercise of a warrant. There is no apparent reason for a similar

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-5- The existence of market imperfections may obviate the discount even in the presence of conditions (1) and (2) above. For example the benefits from economies of scale in transactions costs or information collecting and processing may outweigh the management fee costs to the investors. This might tend to explain the occasional premiums found on closed-end funds. Even in this case the management fees have positive value. The premium arises from the synergy that allows the economic value of the fund to exceed its asset value.
-6- The intrinsic value of a warrant or call option is the current stock price less the striking price.

arbitrage condition to be applicable to the capital shares
since they are not exerciseable except at maturity.
Furthermore, the model presented in the previous sections
exhibits a reasonable fund structure in which a capital
share discount is possible.

Before proceeding to examine the capital discount of
this model in more detail it is worthwhile to point out that
unlike the fund discount the behavior of this discount is
qualitatively dependent upon the assumptions of section II,
in particular the proportionality assumptions (A6) and
(A7).-7- Risk, however, is not a necessary condition for the
discount although it does affect the magnitude. Appendix B
demonstrates that the capital shares of a dual fund that
holds only riskless securities will be subject to a
discount.

As shown in figure 4, the capital discount is a
monotone increasing, concave function of V limited in
value by $1 - e^{-\gamma \tau}$ Below a critical value, V_c, the
discount is negative (ie. a premium). The premium is
unlimited in size. From (7) and (17) the critical value can
be determined in priciple as the solution to

-7- The matter of the discount is identical to the
question of premature exercise of an option addressed by
Merton |4| and Samuelson and Merton |7|. They provide an
example in which an early exercise (and, hence, in our model
a discount) will never arise. The example is a constant
dividend stream smaller than rE. This would be implausible
for a dual fund since it would require that the income
shares were issued to yield less than the riskless rate;
nevertheless, it does demonstrate that capital discounts may
never occur in some cases. It can be readily demonstrated
that a necessary condition for discounts to be impossible on
the capital shares is that the payouts be a bounded function
of V.
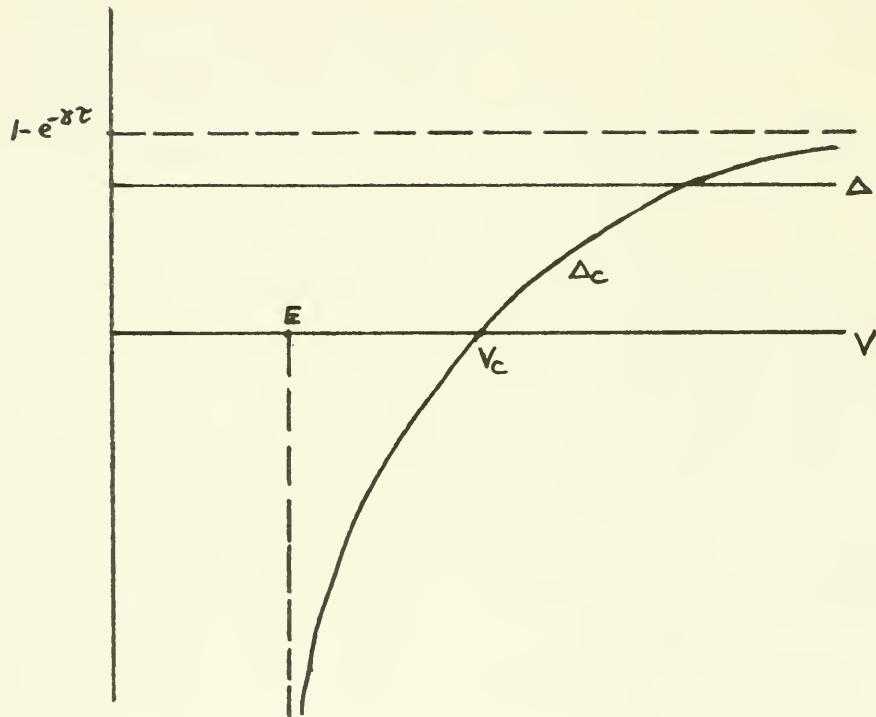
Figure 4

$$(19) \quad V = E \frac{1 - \phi(x_2) e^{-r\tau}}{1 - \phi(x_1) e^{-\gamma\tau}} = E \, \Gamma(V, \tau)$$

Examination of Γ reveals that it is bounded above by $(1 - e^{-\gamma\tau})^{-1}$ hence discounts are possible for funds of any maturity.

When $V \gtrsim rE/\gamma$, the discount is an increasing function of maturity. When $V < rE/\gamma$, the discount decreases with τ for short maturities and increases with τ for long matuirties. Since $\gamma < r$ is the general rule,-8- either case above will be possible depending on the current value of the fund.

In the perfect capital market of this model, concern over the discount is vacuous since it bears a strict functional relationship to the asset value and other variables. Furthermore, since the fund as a whole always sells at a discount (ie. $\Delta > 0$), it is clear from (20) below that either the capital shares or the imcome shares must always be selling at a discount. If both discounts are perceived to be bad then one or the other group of shareholders must be disappointed at all times. If, on the other hand, the discount is recognized to have no importance of its own, then a capital share discount might be recognized as good since it corresponds to a high asset value.

$$(20) \quad \Delta = \frac{V - E}{V} \Delta_c + \frac{E}{V} \Delta_I$$

Leaving discounts we turn our attention to the risk and return characteristics of the fund and its claims. First we

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-8- Aside from the empirical resolution of this statement, this matter has a theoretical content developed later in this section.

shall determine the equilibrium dividend policy. All of the funds will redeem the income shares at the initial offering price; furthermore, each was set up to have the capital and income shares offered at that same price.-9- Ignoring the management fees then, the equilibrium dividend policy is that value of γ for which E will be the price of both shares at a maturity equal to the original lifetime of the fund. If T is this lifetime then setting $f(V,T)=E$ and $V=2E$ in (7), γ^e will be the solution to

$$(21) \qquad 1 = 2e^{-\gamma T}\phi(x_1) - e^{-rT}\phi(x_2)$$

Clearly γ^e is a function of only T, r, and σ^2. By the implicit function theorem

$$\text{a)} \quad 0 < \frac{\partial \gamma^e}{\partial r} = -\frac{f_r}{f_\gamma} < 1$$

$$(22) \qquad \text{b)} \quad 0 < \frac{\partial \gamma^e}{\partial \sigma^2} = -\frac{f_{\sigma^2}}{f_\gamma}$$

$$\text{c)} \quad 0 \lessgtr \frac{\partial \gamma^e}{\partial T} = -\frac{f_T}{f_\gamma} \qquad \text{as } f_T \gtrless 0$$

where $f_T \equiv f_\tau|_{\tau=T}$. Also

$$\text{a)} \quad \gamma^e(T,r,\infty) = (\log 2)/T = \bar{\gamma} \qquad \gamma^e(T,r,0) = r - \frac{1}{T}\log\left[\frac{1+e^{rT}}{2}\right]$$

$$\text{b)} \quad \gamma^e(T,\infty,\sigma^2) = (\log 2)/T$$

$$\text{c)} \quad \gamma^e(\infty,r,\sigma^2) = 0 \qquad\qquad \gamma^e(0,r,\sigma^2) = \begin{cases} r/2 & \sigma^2 = 0 \\ 0 & \sigma^2 > 0 \end{cases}$$

-9- For Putnam the capital share issue price was half that of the income shares; however, twice as many were offered. In all cases the initial leverage was fifty-fifty.

Since f is homogeneous of degree zero in $r, \gamma, \sigma^2,$
and $1/T$, it follows that γ^e is homogeneous of degree one
in the other parameters. Hence for computational ease it
may be written $g(rT, \sigma^2 T)/T$ and only two variables,
lifetime uncertainty and riskless return are of concern. The
partial derivatives in (22) have the expected signs. The
income share owners must be promised more when the interest
rate is higher, when the fund is riskier, or when a change
in maturity would hurt them. Note, however, that an
increase in the interest rate is not fully reflected in
as the spread between γ^e and r decreases. Furthermore,
the larger is r or σ^2 , the lesser is the part that the
other plays in determining the equilibrium dividend yield.
Indeed in the limit γ^e is independent of the other as (23)
shows.

The importance of γ^e should not be over emphasized.
It is the equilibrium policy only under the postulated
offering and redemption format. While this is the format of
all the existing dual funds and, given the legal restriction
of a maximum two to one leverage, it is likely to be the
chosen format of any new fund, alternate offering and
redemption structures would lead to other equilibrium
dividend policies.

If the management fees are not neglected, then the
determination of the equilibrium dividend policy δ^e and fee
percentage μ^e becomes a simultaneous problem. The two
equations to be satisfied are $P_c = f(V, T)$ and $P_i = F(V, T)$,
where the P's denote the offering prices of the shares. The
first equation is derived from (7) and will be similar to

(21). The latter is from (11). This proceedure, however, leads only to the trivial solution $\mu^e = 0$, $\delta^e = \gamma^e$ in the usual case. Since the management puts up no front money; ie. it does not bid for the right to serve as the management and collect the fees, we have $P_C + P_1 = \hat{f} = V$. From (12) the only solution set is clearly the one given above. All the above is, of course, merely a restatement of the fact that closed-end funds with management fees will always sell at a discount. Under the perfect market assumptions in A1, no one would be willing to buy the initial offering of such a fund since he could costlessly perform the same service that for himself and avoid the immediate loss of value that the discount would bring about. As is discussed in footnote 5, the existence of market imperfections may allow a premium on the fund. In this case an equilibrium solution with $\mu^e > 0$ will be possible.

We have stated earlier that $r > \gamma$ is the general case. While this might be considered a purely empirical matter, in actuality it has more content. From (22) and (23) it is clear that γ is bounded by $\overline{\gamma}$. Thus γ can exceed r only if $rT < \log(2)$. For dual funds with fifteen-year lifetimes, this would require an interest rate below 5%. For the small variance rates actually observed the true interest rate or lifetime would have to be markedly smaller.

Once δ^e has been determined, the dividend structure of returns is established since the dividend policy, unlike the dividends themselves, is nonstochastic by assumption. The structure of returns from capital gains remains to be determined.

From the derivation of the pricing equation (see Appendix A or Merton $|5|$) it is clear that the arbitrage condition

$$(24) \qquad \frac{\alpha_f - r}{\alpha - r} = \frac{\sigma_f}{\sigma} = \frac{Vf_v}{f} \equiv q$$

holds at every instant of time. A similar relationship, of course, obtains for the income shares. We denote this ratio of relative excess returns and relative risk by q for the capital shares and Q for the income shares. Z, defined in (7), and $U \equiv \sigma^2 \tau$ are sufficient statistics for the former; the latter is described fully by $V, \sigma^2,$ and $\tau.$

The figures on the next page depict the behaviour of q and Q. As $U \to o$ the final resolution of the claims becomes certain either because $\sigma^2 \to o$ and the uncertainty is removed or $\tau \to o$ and the resolution is imminent. If $Z > 1$, then the fund will almost certainly be able to repay the income share holders. Hence these shares become riskless $(Q=0, \alpha_f = r)$, and the capital shares take on the levered asset risk $(q=e^{-\delta \tau}V/f)$. If $Z < 1$, then the fund will almost certainly default. The capital shares become worthless and infinitely risky; the income shares acquire the entire asset risk $(Q=1, \alpha_f = \alpha)$

The response of q to Z in figure 5a is as expected; however, that of Q to V deserves some explanation. If we consider the dual nature of the income share as we did in section III, then the risk of the redemption claim, Q^H, is identical to that of a discount bond $|5|$; thus it is a decreasing function of V. The risk of the dividend claim,
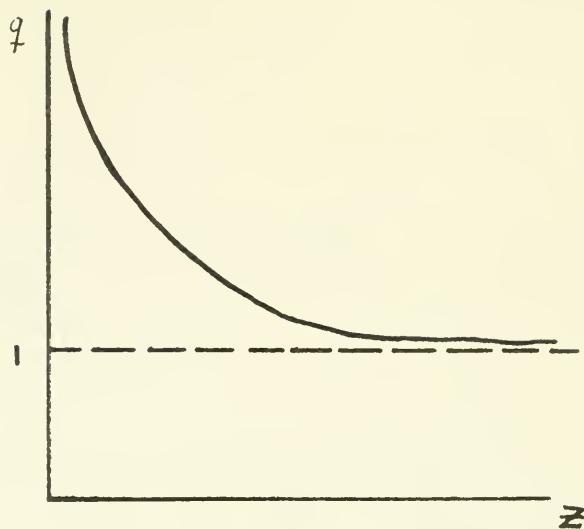
Figure 5a
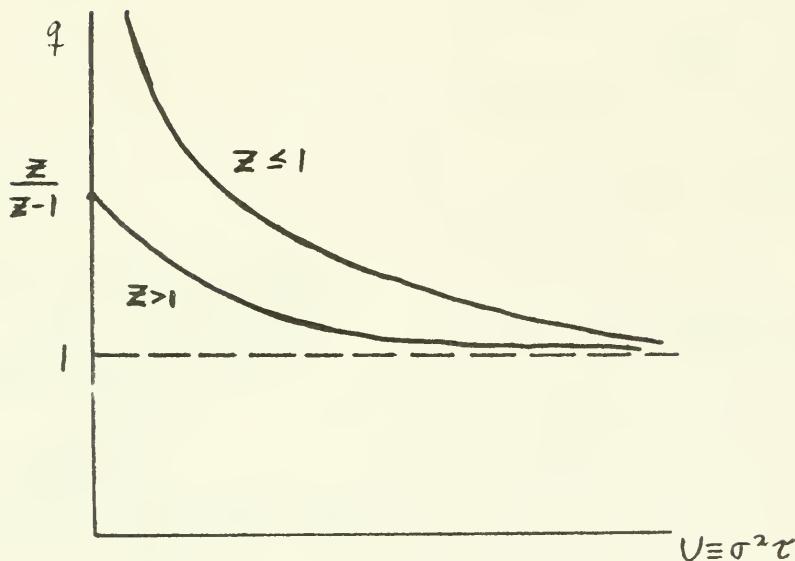
Figure 5b
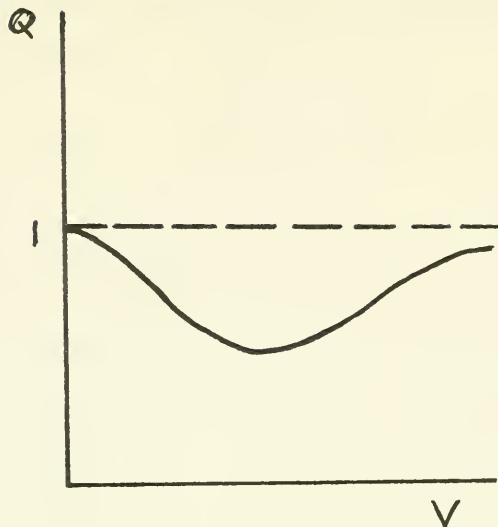
Figure 5c
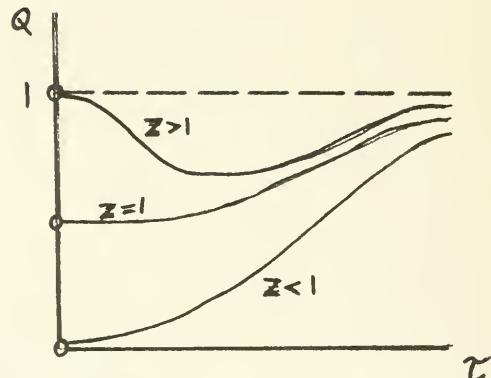
Figure 5d
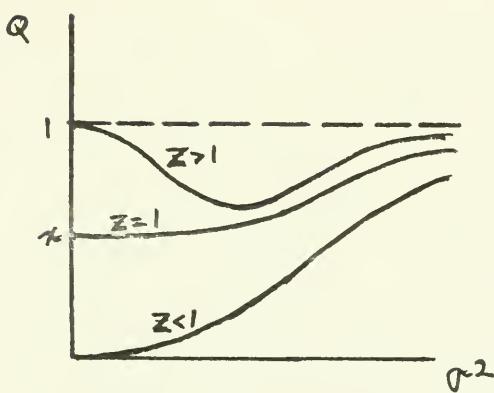
Figure 5e

$$x = \frac{1}{2} + \left[2 + \frac{1 + \frac{1}{z}}{e^{-\delta \tau} - 1} \cdot \frac{\gamma}{\delta}\right]^{-1}$$

$$\frac{1}{2} \le x \le 1$$

Q^h , is unity. Q is a weighted average of these two with weights equal to their proportional to the sub-claims' values.

$$(25) \qquad Q = \frac{h}{F} Q^h + \frac{H}{F} Q^H = \frac{h}{F} + \frac{H}{F} Q^H$$

Initially aS V increases Q^H falls and takes Q with it; however, as V tends to infinity the weight H/F goes to zero and Q rises toward one again. The same process is at work in figures 5d and 5e. Both figures are similar to figure 8 in |5| although they are distorted in the same fashion as is figure 5c since h/F approaches one as σ^2 or τ becomes large. Summarizing the comparative statics:

$$
\begin{array}{ll}
q_z < 0 & q_U < 0 \\
Q_V \gtrless 0 \\
Q_\tau > 0 , \ z \leq 1 & Q_\tau \gtrless 0 \quad z > 1 \\
Q_{\sigma^2} > 0 , \ z \leq 1 & Q_{\sigma^2} \gtrless 0 \quad z > 1
\end{array}
$$

(26)

The ratio q is a concept similar to what is popularly known as capital share leverage. The term leverage is used here in the same sense that it would be applied to the equity of any company with debt in its capital structure. There are two common uses which we will distinguish as structural and effective leverage. The "structural leverage" of the capital shares is defined as $L = V/(V-E)$. The "effective leverage" is $\lambda = V/f$. Both of these are intended to measure the number of dollars working for the capital shares per dollar invested although clearly

effective leverage being based on market rather than book values does so more accurately. However, this does not imply that it is necessarily more meaningful as a measurement of risk and return. From its definition and that of q, the effective leverage is always greater than q since $\lambda = q/f_v > q$. Since $L = (1-\Delta_c)\lambda$, when the capital shares are selling at a discount, the structural leverage will be less than the effective leverage. In this case the former might well be a better estimate of the risk-return ratio q.

V The Sample Data

The model as developed om the previous sections was
tested on the seven dual purpose funds, American DualVest,
Gemini, Hemishpere, Income & Capital, Leverage Fund, Putnam
Dou-Fund, and Scudder Dou-Vest. The time period examined
extended from May 1967, near the time all seven were
established, through December 1973. The data on capital
share price and the net asset value per share was taken from
the weekly reports of the Lipper Analytical Division of
Steiner Rouse and Co. The former was checked against the
ISL Daily Price Index where the income share prices were
also obtained. Dividend data was found in Moody's Dividend
Record. Expenses and Fees were obtained from Moody's Bank
and Finance Manuals. A time series for asset value per
share was constructed as the sum of net asset value, par
value of income share, -10- dividend arrearage, and
accumulated dividends.

Accumulated dividends were not directly observable
hence they were approximated as follows. If the minimum
dividend was met in each quarter of the year, gross
accumulated dividends were assumed to have to accrued over
the year at a constant rate equal to the total dividends
divided by the number of weeks in the year. If less than
the minimum dividend was paid in any quarter of the year,

-10- In the case of American and Income & Capital the par
value was adjusted annually to reflect the accounting
procedure used to amortize the difference between the net
money received by the dual fund per income share at offering
and the redemption price.

the accrual was assumed to have been at the rate paid quarter by quarter for each quarter prior to that quarter. For the remainder of the year the previous method was applied to the remaining dividends. Net accumulated dividends were figured as the maximum of zero and gross accumulated dividends less dividends actually paid.

This method should closely approximate the actual asset value of the dual fund; however, since dividends actually accrued sporaticly the approximation may produce a time series whose growth rate displays too small a variance.

Market returns were computed weekly as $R'_t = (V'_t + D'_t)/V'_{t-1}$ where V'_t denotes market value at time t and D'_t denotes the value of any dividend that went "ex" during week t. Value returns were computed as $R_t = (V_t + D_t)/V_{t-1}$ where V_t denotes the constructed asset value at time t and D_t the value of any dividend paid during week t. Market returns on the capital shares were computed as the ratio of market prices of the capital shares in two successive weeks.

The data was first checked against the various assumptions in section II. The log of returns was regressed against a constant and the Durbin-Watson statistic was examined to test for serial correlation in dz. The results are presented in table 1. Positive serial correlation does seem to be evident in the value returns series altough this result might also be attributable to the construction process. The other series seem to be free from serial correlation; therefore, we will assume that the construction process is at fault.

Table 1

Summary Statistics of Dual Fund Data

		$\bar{X}$	α	σ^2	D.W.	Skew.	Kurt.
	M	-0.046	-0.027	0.037	2.019	0.306*	3.958*
A	A	-0.042	-0.034	0.016	1.380*	-0.388*	3.390
	C	-0.167	-0.100	0.135	2.212	0.306*	5.632*
	M	0.018	0.036	0.034	1.936	0.163	3.617*
G	A	-0.003	0.009	0.024	2.117	-0.252*	1.582*
	C	-0.054	-0.012	0.083	1.943	0.119	4.224*
	M	-0.146	-0.120	0.053	2.211	0.639*	5.335*
H	A	-0.096	-0.084	0.024	1.806	-0.634*	1.308*
	C	-0.371	-0.272	0.199	2.346*	0.545*	4.024 *
	M	0.001	0.022	0.042	2.091	0.086	3.314
I	A	-0.003	0.008	0.022	1.556*	-0.190	3.242
	C	-0.072	-0.229	0.098	2.110	0.180	3.584
	M	-0.031	-0.016	0.030	2.056	0.216	3.120
L	A	-0.017	-0.007	0.021	1.917	-0.191	7.464*
	C	-0.106	-0.069	0.075	2.010	0.090	3.758*
	M	0.022	0.039	0.027	1.823	-0.130	5.675*
P	A	0.009	0.017	0.015	1.483*	-0.185	3.377
	C	-0.119	-0.068	0.104	1.941	0.517*	4.236*
	M	-0.028	-0.012	0.033	2.116	0.041	4.037*
S	A	-0.010	-0.004	0.011	1.546*	-0.246	2.901
	C	-0.063	-0.218	0.083	2.162	0.069	3.715*

* significant at the 1% level

$$\bar{X} = \frac{1}{N} \sum \log(R_t)$$

$$\sigma^2 = \text{Var}\left[\log(R_t)\right]$$

$\alpha = \bar{X} + \frac{1}{2}\sigma^2$, the instantaneous expected rate of return

M, A, and C denote the market value, constructed asset value, and
capital shares respectively

Standard skewness and kurtosis tests were also performed on the log-returns series. These results are also presented in table 1. The market series seem to demonstrate positive skewness while the value series show negative skewness. This effect is most pronounced for American and Hemisphere. Lepto-kurtosis also present in many of the series suggests that the log-returns are not normally distributed. These results are not surprising since these tests are highly sensitive to even small deviations from normality in samples of this size. Two alternate explanations for lepto-kurtosis are a stable distribution with infinite varaince |3| or a normal distribution with variable variance |6|.

In the former case the sample variance will not be a reliable estimate of the dispersion of the log-returns, and our estimation procedure using σ^2 may give poor results. In the latter case the sample variance will be a good estimate of the contemporaneous population variance; however, it may not be the best estimate of the future prevailing variance. In either case the derivation in section II is no longer rigorously valid since we have assumed that dz is a Gauss-Wiener process, A.3, and that the variance is non-stochastic, A.3, and constant, A.5.

The proportionality of dividends and fees was checked by regressing dividends and per share fees and expenses -11- against average value during each year. These results are

----

-11- Expenses were included with management fees as a part of cash flow on the presumption that the shareholders expected to pay them in the future even though they were not contracttually obligated to do so as with the managemnt

Table 2

Results of the Dividend and Expenses
Proportionality Tests

Dividends

	Equation	Correlation	t-stat(a)	$\overline{\delta}$ (%)
A	.57 + .015 V	.35	1.15	3.72
G	-.27 + .046 V	.59	0.33	3.61
H	.51 + .006 V	.35	3.77*	3.59
I	.49 + .014 V	.26	1.13	3.62
L	.67 + .009 V	.32	1.57	3.30
P	1.64 - .011 V	-.46	4.60*	3.50
S	.70 - .005 V	-.38	5.72*	3.31

Expenses

	Equation	Correlation	t-stat(a)	$\overline{\mu}$ (%)
A	.12 + .0002 V	.32	16.05*	0.97
G	-.02 + .0040 V	.71	0.36	0.85
H	.05 + .0010 V	.62	4.29*	0.93
I	.13 - .0020 V	-.30	1.81	0.75
L	.03 + .0009 V	.29	0.77	0.71
P	.09 + .0052 V	.66	0.97	0.76
S	.08 - .0021 V	-.43	1.88	0.72

* significant at the 1% level

presented in table 2. The intercept terms were significantly positive for Hemisphere's, Putnam's, and Scudder's dividends and for American's and Hemisphere's expenses. For the others the hypothesis of proportionality can not be rejected; however, the small sample size of only seven years precluded a powerful test.-12-

The correlations for all of the regressions were low, and in a few cases negative. The negative slope for Putnam and Scudder is probably due to the management's reluctance to pay dividends below the minimum. Currently Putnam is faced with a rising minimum dividend and Scudder with meeting an already existing arrearage at a time when the values of all the funds are low. Consequently, we might expect the managers to shift the portfolios to income producing securities and therefore paying larger dividends than they did earlier when the fund value was higher.

----

fees.
-12- Square and higher order terms could not be added to the regression equation without drastically reducing the degress of freedom.

VI The Model

Model values for the capital shares were constructed
from the asset values and the market values of the dual
funds. -13- All model parameters were estimated using only
historic data except during 1967 for which that whole year's
data was used. The variance rates were estimated by the
sample variances of the appropriate log return series. This
parameter was updated weekly. The cash flow proportionality
constant was estimated by the historic mean of C/V where V
represents the average value of V during the year. This
parameter was updated yearly. The risk-free rate chosen was
the median yield-to-maturity on all taxable government bonds
with maturities in excess of fifteen years.

Table 3 and figures 6 through 12 present statistics and
plots of the models. The plotted points represent an
average value over a period of four weeks.

Both models capture the behavior of the capital shares
quite well; however, the model based on market prices seems
better. Negative bias was present in both model types for
all based on market price seems better. In particular the
residual error, left after the effects of possible
misspecification are removed, is smaller. Negative bias was
present in both model types for all of the dual funds.
Furthermore, this bias appears to be an increasing function

-13- The pricing formulas developed earlier were based on
asset value rather than market value; however, the
derivation goes through exactly the same if market value is
independent of the leverage on the dual fund. The market
model may be better if the constructed asset values are in
error or if expenses are not proportional to asset values.

Table 3

Comparison of Model and Market Values

	A	G	H	I	L	P	S
Market Based Model							
R^2	0.981	0.695	0.987	0.927	0.889	0.967	0.821
RMSE	2.206	2.717	1.464	2.059	2.516	1.046	0.911
Mean error	2.107	2.424	1.309	1.788	2.450	0.949	0.790
Estimated slope	1.188	1.061	1.228	1.404	1.243	1.166	0.818
Fraction of Error due to:							
Bias	0.912	0.796	0.799	0.754	0.948	0.822	0.751
$b \neq 1$	0.049	0.001	0.146	0.126	0.012	0.066	0.046
Residual variance	0.039	0.203	0.055	0.120	0.040	0.112	0.203
Misspec. Error	4.676	5.883	2.024	3.730	6.076	0.971	0.661
Residual Error	0.189	1.498	0.117	0.508	0.253	0.122	0.168

	A	G	H	I	L	P	S
Asset Based Model							
R^2	0.870	0.477	0.908	0.342	0.185	0.710	0.268
RMSE	1.802	3.617	1.511	2.731	1.845	1.132	1.039
Mean Error	1.337	3.203	0.542	1.693	0.254	0.199	-0.173
Estimated slope	1.173	0.750	1.587	1.012	0.346	1.350	0.456
Fraction of Error due to:							
Bias	0.551	0.784	0.129	0.384	0.019	0.031	0.028
$b \neq 1$	0.057	0.020	0.500	0.000	0.439	0.137	0.333
Residual variance	0.392	0.196	0.371	0.616	0.542	0.832	0.639
Misspec. Error	1.974	10.517	1.435	2.863	1.558	0.214	0.389
Residual Error	0.185	2.564	0.847	4.594	1.844	1.066	0.440

Note: For an explanation of the items in this table
please see Appendix C

FIGURE 6

ERICAN DUALVEST

-     Capital Share Market Price

x   Market Value Based Model

o   Asset Value Based Model

FIGURE 7

GEMINI FUND

- Capital Share Market Price

x Market Value Based Model

FIGURE 8

ISPHERE FUND

- Capital Share Market Price
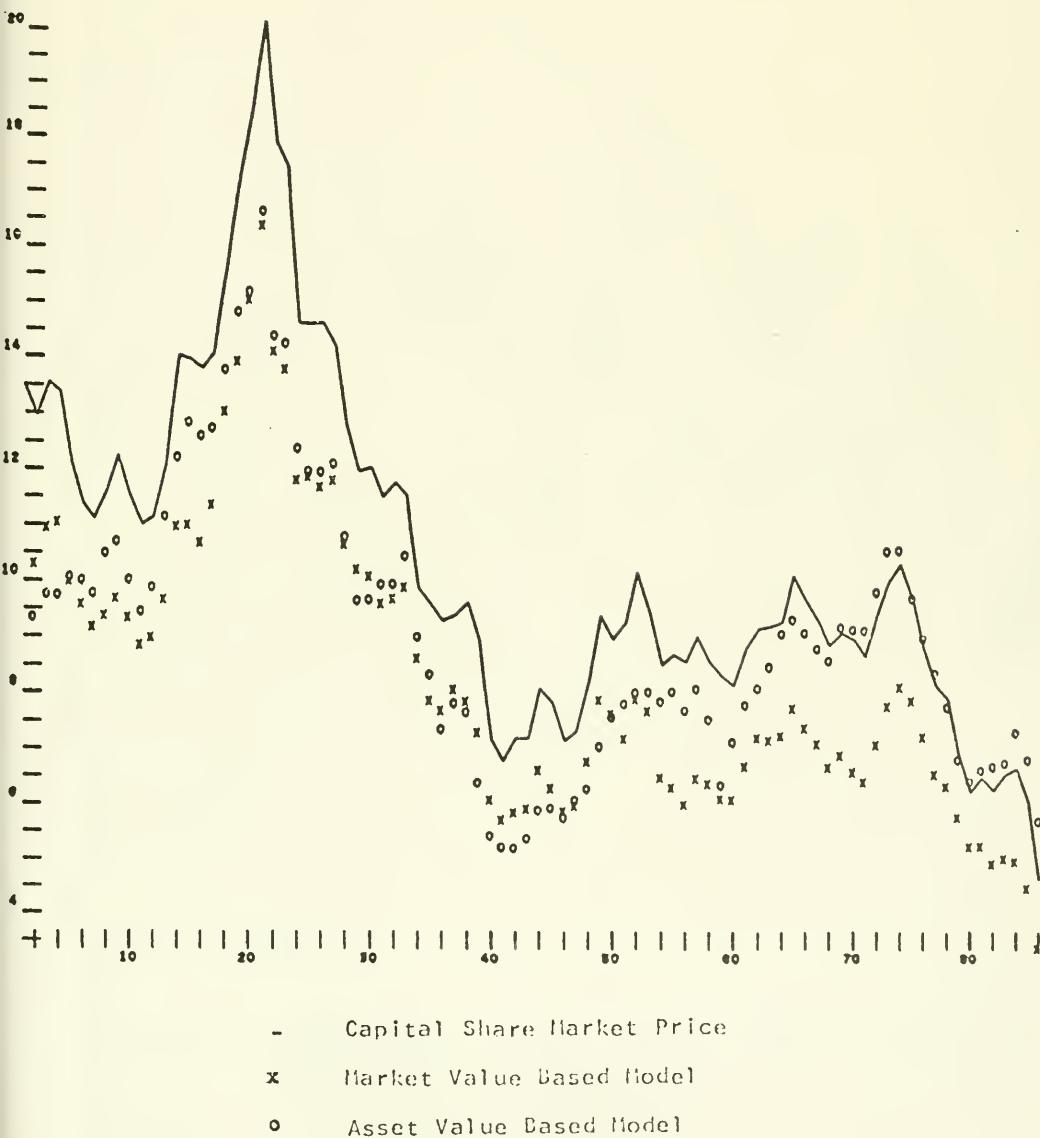
x Market Value Based Model
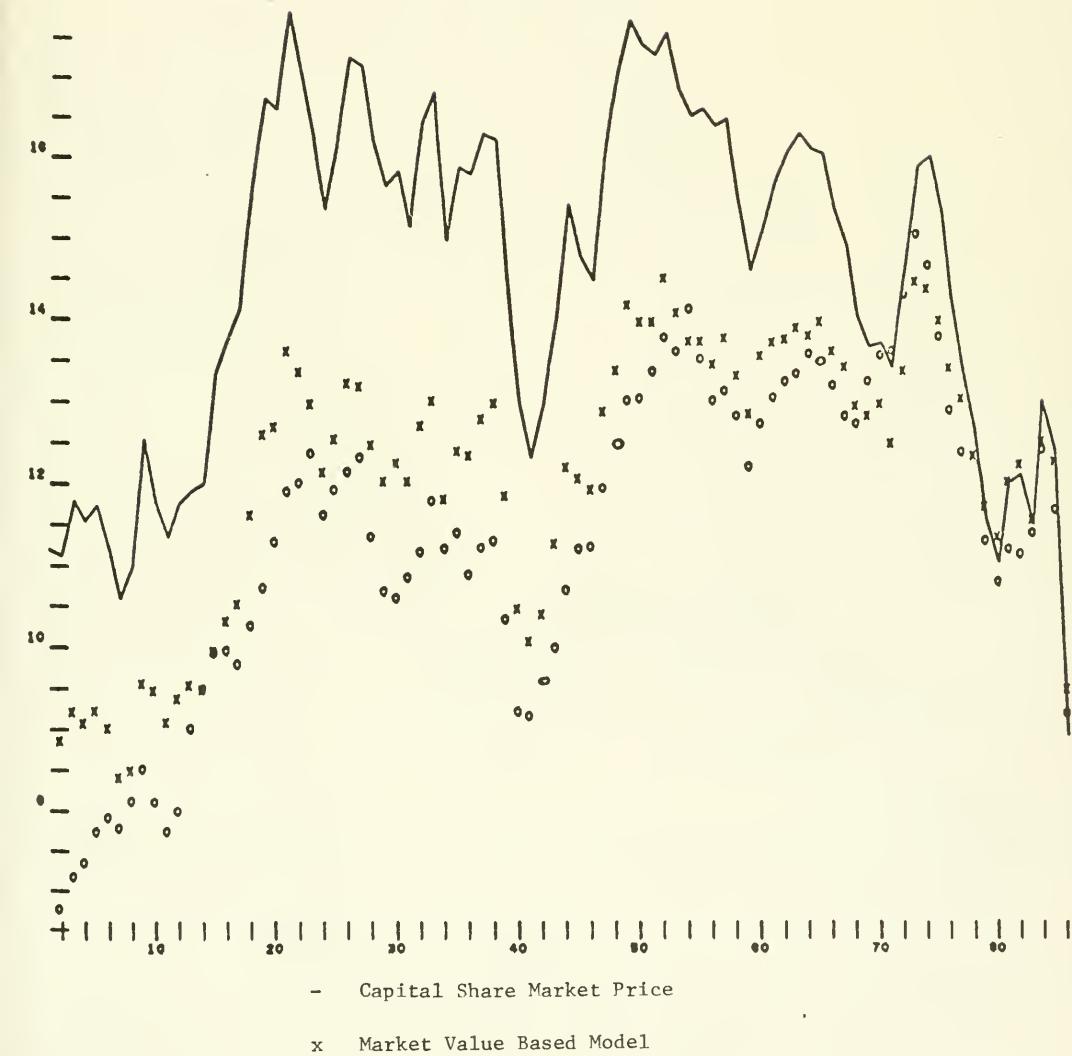
o Asset Value Based Model
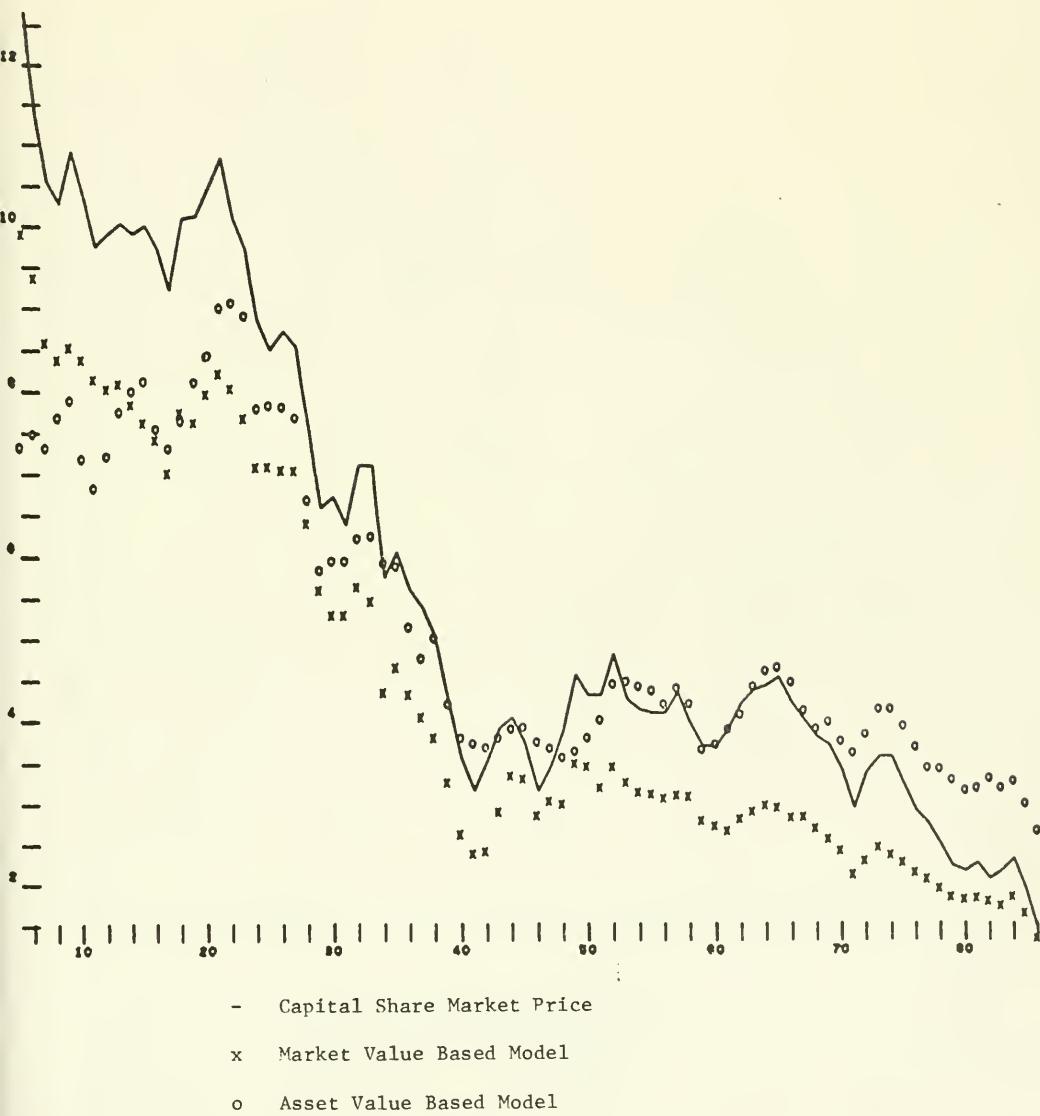
FIGURE 9

INCOME & CAPITAL SHARES

-    Capital Share Market Price

x   Market Value Based Model

o   Asset Value Based Model

FIGURE 10

LEVERAGE FUND

- Capital Share Market Price

x Market Value Based Model
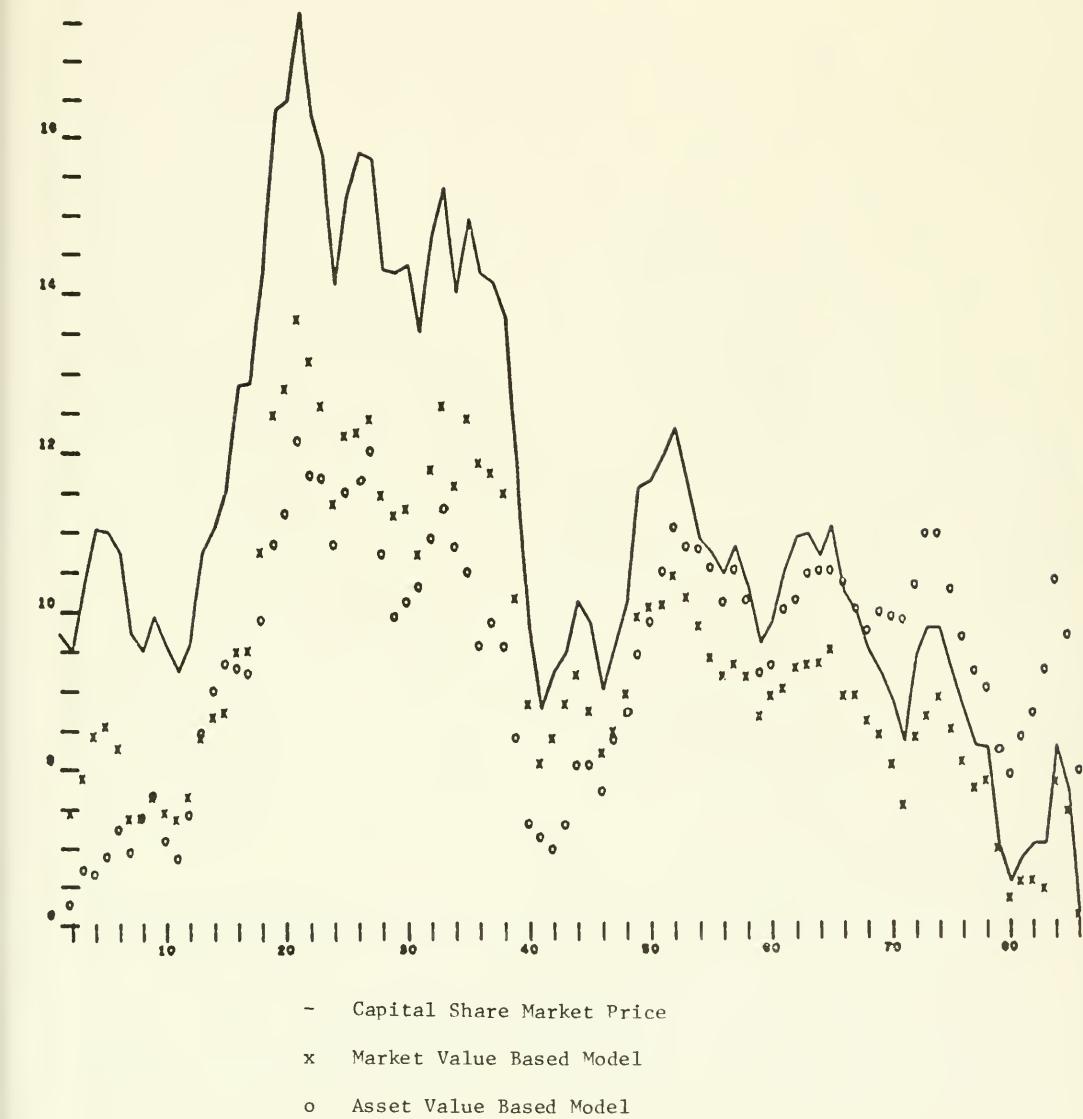
o Asset Value Based Model
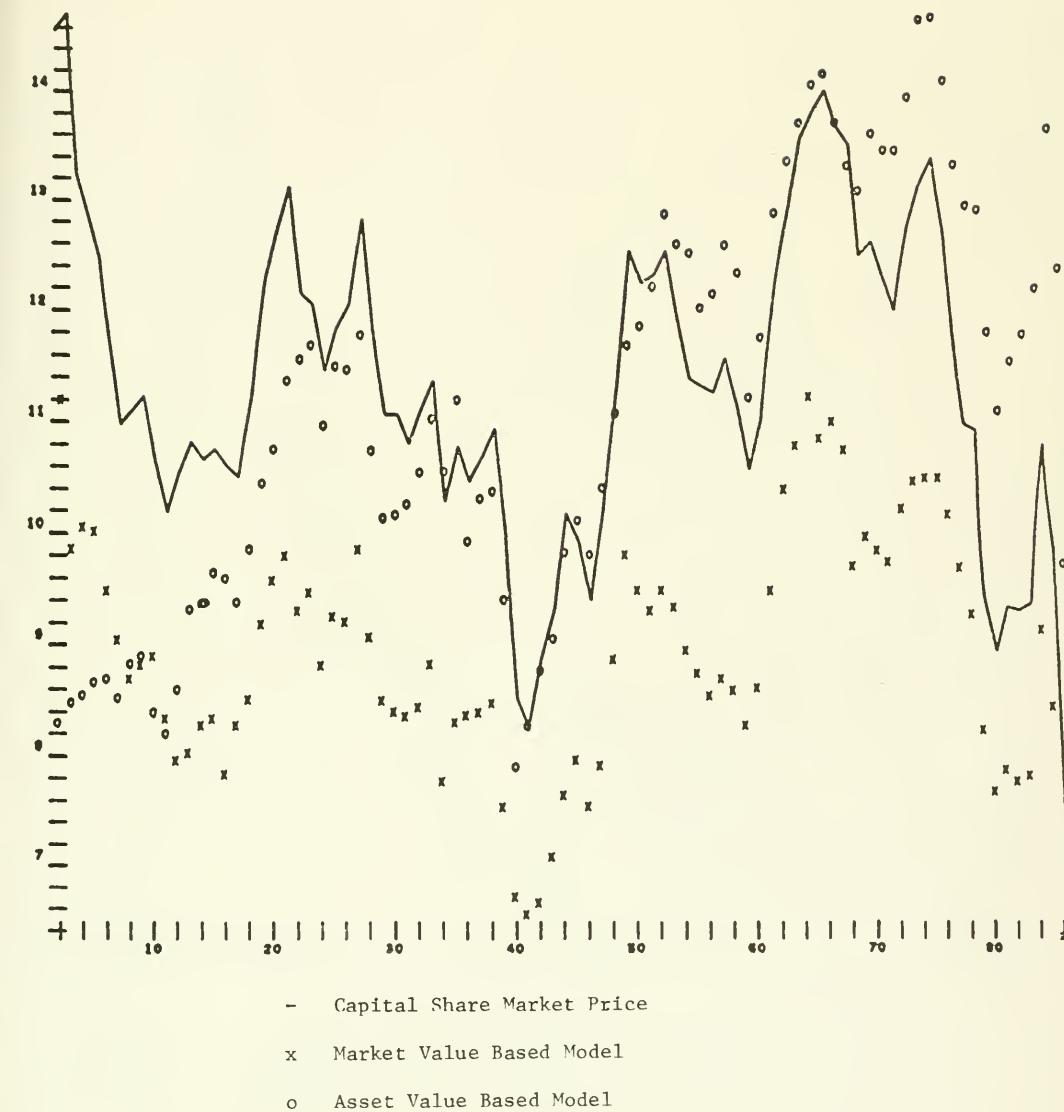
FIGURE 11

'NAM DOUFUND

-    Capital Share Market Price

x   Market Value Based Model

o   Asset Value Based Model

FIGURE 12

DER DOU-VEST
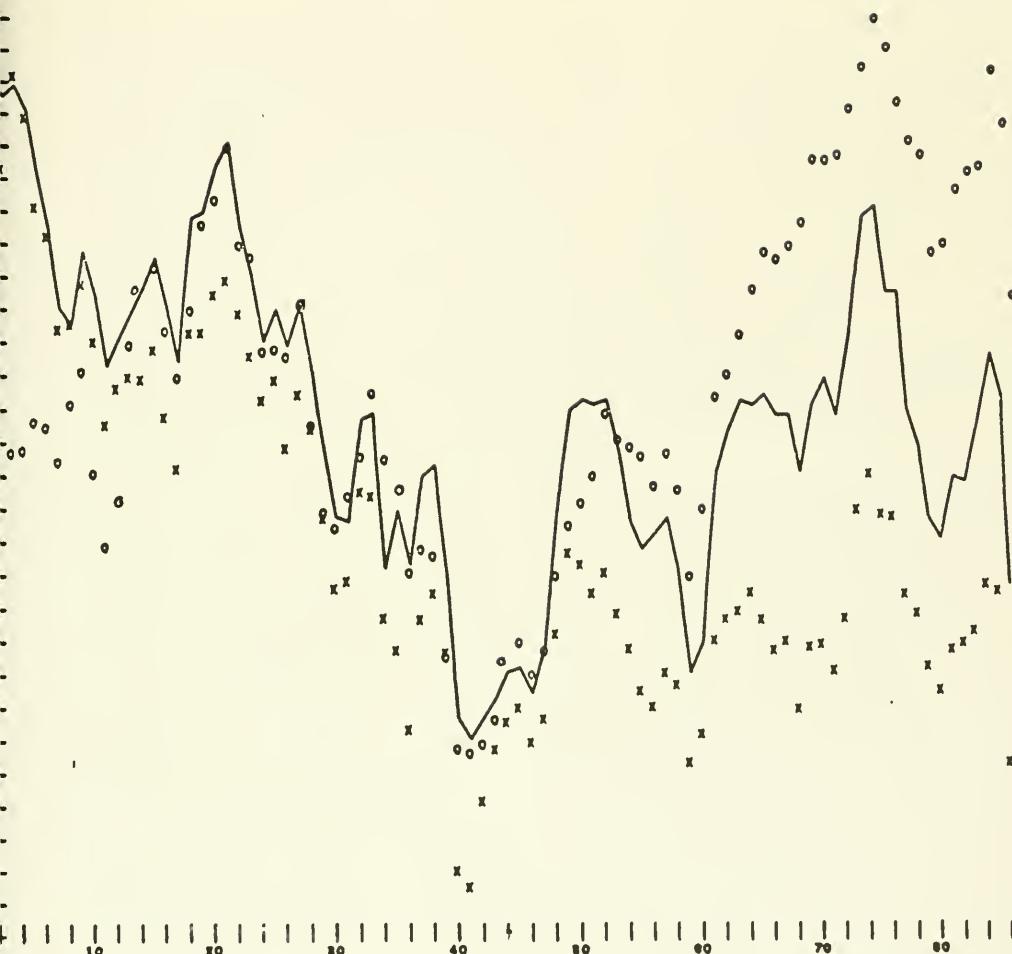
- Capital Share Market Price
x Market Value Based Model
o Asset Value Based Model

of maturity. To verify this qualitative result the
following regression was performed.

$$\log\left[\left(\tfrac{market}{model}\right)_t\right] = a + b \cdot \log\left[\left(\tfrac{market}{model}\right)_{t-1}\right] + \varepsilon_t$$

The results shown in table 4 confirm the hypothesis that the
error decreases with time since b is significantly less
than one for most cases.-14- Two alternative explanations
for this behavior in the discrepancy are: (1) The model is
misspecified in a manner that introduces bias which is an
increasing function of maturity, or (2) the market
participants do not fully utilize the information contained
in past estimates of the variance or other parameters in
forming their investment decisions.

If the market price is "incorrect" (ie. explanation (2)
is at least partially valid), then the model will indicate
opportunities for abnormal profits by buying the undervalued
and selling the overvalued capital shares. Furthermore,
this opportunity will have little associated risk since we
know how to hedge the capital shares with positions in the
income shares and bonds.

To test whether the market-model discrepancy is due in
part to the market's inefficiency in utilizing any
information contained in the model in pricing the capital

-14- Care must be taken in interpreting these results
since the Durbin Watson statistic indicates the presence of
auto-correlation. If ρ is the auto-correlation coefficient
then $\hat{b}$ is an inconsistent estimate of b (plim $\hat{b}$ = b +
$(1-b^2)/(1+b\rho)$.) and in our regression $\hat{b}$ is biased away
from 1. The usual correction proceedure leads to a
regression of the form $y_t - (\rho + b)y_{t-1} - \rho b y_{t-2}$ which is not
identified for ρ and b.

Table 4

Results of Regression:

$$\log(mkt(t)/model(t)) = a + b \log(mkt(t-1)/model(t-1))$$

	A	G	H	I	L	P	S
	---	---	---	---	---	---	---
				Market Based Model			
a	0.031	0.002	0.031	0.007	0.039	0.027	0.007
b	0.870	0.986	0.884	0.955	0.839	0.834	0.939
s.e. (b)	0.028	0.011	0.026	0.015	0.030	0.030	0.020
t-stat (b=0)	30.637	87.436	33.537	63.547	28.254	27.811	47.391
t-stat (b=1)	4.597	1.257	4.403	2.971	5.421	5.548	3.104
D.W.	2.335	2.534	2.341	2.510	2.245	1.967	2.440
				Asset Based Model			
a	0.007	0.008	-0.004	0.001	-0.001	-0.001	-0.003
b	0.936	0.963	0.960	0.982	0.965	0.958	0.961
s.e. (b)	0.019	0.015	0.015	0.010	0.012	0.015	0.015
t-stat (b=0)	48.146	65.596	63.618	99.623	78.147	64.659	66.240
t-stat (b=1)	3.319	2.498	2.672	1.832	2.863	2.822	2.701
D.W.	2.372	2.487	2.612	2.493	2.733	2.251	2.494

and income shares, the following simulation test was
performed. Each week the model value for the capital share
was compared to the existing market price. If the former
was larger (smaller), one dollar was invested in a long
(short) position in the capital share. This investment was
financed by an opposite position in the dual fund (ie. an
equal number of capital and income shares) and borrowing or
lending at the riskless rate. The aggregate investment was
constrained to be zero. Furthermore, the relative positions
were taken so to form a hedged portfolio that would minimize
the risk.

Under the idealized conditions of the model, the proper
hedge is $-Vf_v/f$ "dollars" in the fund for each "dollar" in
the capital shares. Care must be taken here since "dollar"
refers to investment at the true price. If the model rather
than the market reflets the correct price, the the proper
amount to invest in the fund is $-Vf_v/P_c$ (where P_c is the
market price of all the capital shares).-15-

In the test performed the returns computed were not
riskless. Variations in the returns would be expected from
three sources: (1) weekly rather than continuous portfolio
updating, (2) change in the market-model deviation, and (3)
improper hedges if the model price were not correct. The
second source is that from which profits are to be made.
The other two will introduce noise into the return series.
To reduce the noise a combination portfolio was also formed.

-15- If V and f are the true prices then $-f_v$ shares
of the fund must be held to hedge one capital share; hence,
$-f_v V/P_c$ dollars must be invested in the fund per dollar in
the capital shares.

In this portfolio one-seventh dollar was invested in each capital share and the seven hedges were also utilized. The combination portfolio will show improved results to the extent that the noise represents unsystematic risk.

Table 5 presents the results of this market simualtion. The top line for each period is the mean weekly return on the hedging portfolios in cents. The lower lines give the standard deviation of return and the t-statistics. The mean weekly returns are positive in just over sixty percent of the cases and are positive for each fund over the six year period. However, the confidence level on the overall mean return of 0.102 cents is only 9%. From these results we should be hesitant about rejecting the hypothesis that the market is efficient with regard to the model information. Hence our preliminary conclusion must be that any model-market discrepancy must be due to errors in the model.

TABLE 5

RESULTS OF MARKET SIMULATION TEST

	A	G	H	I	L	P	S	AVG
1968	-0.164 (2.00) -0.59	-0.13 (1.96) -0.48	-0.012 (2.32) -0.04	-0.212 (1.94) -0.79	-0.014 (2.47) -0.04	-0.404 (3.07) -0.93	-0.052 (2.83) -0.20	-0.142 (2.25) 1.19
1969	0.354 (2.29) 1.12	0.021 (1.62) 1.0	0.047 (2.27) 0.10	0.151 (1.65) 1.63	0.048 (2.22) 1.16	0.459 (3.0) 0.86	1.15 (2.15) 0.37	0.148 (2.50) 1.41
1970	0.135 (2.99) 0.3	0.042 (2.00) 0.15	0.340 (5.42) 0.45	0.205 (1.0) 0.57	0.068 (2.54) 0.19	0.762 (5.97) 0.44	0.52 (5.72) 1.19	0.254 (3.82) 1.25
1971	-0.092 (2.85) 1.23	0.82 (1.0) 0.0	-0.161 (4.02) 0.2	0.006 (1.0) 0.02	0.09 (1.00) 0.3	0.021 (3.0) 0.04	0.023 (2.0) 1.0	-0.007 (2.80) -0.02
1972	0.045 (2.15) 0.1	0.50 (1.0) 0.30	0.030 (3.78) 0.06	0.072 (1.0) 0.28	0.013 (1.0) 0.05	0.253 (3.33) 0.54	0.200 (2.94) 1.0	-0.035 (2.41) -0.27
1973	0.012 (2.73) 0.3	0.25 (2.0) 1.3	0.960 (4.99) 1.75	0.523 (2.53) 1.5	0.242 (2.72) 1.0	0.278 (2.55) 0.55	0.134 (1.0) 0.55	.354 (2.84) 2.35
	0.057 (2.54) 0.40	0.071 (1.81) 0.68	0.167 (3.95) 0.75	0.164 (2.08) 1.38	0.057 (2.25) 0.0	0.084 (4.03) 0.36	0.05 (2.29) 0.70	0.102 (2.83) 1.57

In this section we shall examine possible sources of
model error. In particular inaccuracies that might produce
an error which would be an increasing function of maturity
will be sought. We shall confine our attention to the market
value based model since it had a smaller residual error.
Changes examined here should mostly effect the
misspecification error.

Positive measurement error in the proportionality
constant is one possible explanation. Since   occurs only
in the factor e   in the first argument of W and since
W   > 0, it is clear that overestimation of   will lead to
an underestimation of the capital share values; furthermore
this error will increase with maturity. Similarly a
negative measurement error in the variance is a possible
explanation. To see this we write f = Ee   W(Z,U;1,0,1)
where U =   and note that W > 0. However, either of
these explanations would require a measurement error in the
parameter of the same sign for all the dual funds during
each year.-16-

An alternate explanation that appears more plausible is
that the bias effect is due to ignoring the stochastic

-16- Even if the sample variance is an unbiased estimate
of the true variance, it does not follow that f is an
unbiased estimate of the capital share price. f is neither
a strictly concave nor convex function of variance, thus
Jensen's inequality is inapplicable and the sign of the bias
depends on the current value of V. To test for measurement
error, the model was estimated using all the available data
to estiamte γ and σ^2. No improvement, and little change
at all, was found in this new modeled price series.

nature of the riskless rate. Appendix A prices the capital shares under the assumption of a stochastic interest rate. The solution from (A10) is:

$$(27) \qquad f^*(V, P, \tau) = E P(\tau) W(Z^*, U^*; 1, 0, 1)$$

$$where \qquad Z^* \equiv \frac{V e^{-\gamma \tau}}{E P(\tau)}$$

$$U^* \equiv \int_0^\tau (\sigma^2 + \nu^2 - 2 \rho \gamma \sigma) ds$$

$P(\tau)$ is the price today of a riskless dollar at τ, ν^2 is the instantaneous variance of return on the bond, ρ is the instantaneous correlation coefficient of returns on the bond and the asset value, and W is the option solution defined earlier. Therefore, if σ^2 is assumed constant as before and $\nu^2 - 2 \rho \sigma \nu > 0$, then the earlier model will underestimate the capital share values by an error that increases with maturity.

The parameters ν and ρ cannot be easily determined from a time series of P since they must be functions of maturity if the unanticipated returns to bonds are to be serially independent.-17- If $R(\tau)$ is the yield to maturity on a τ period discount bond (ie. $P = e^{-R\tau}$) and is assumed to be approximately equal to the rate on bonds with maturities close to τ, then the dynamics for R are $dR = a(R)dt + b(R)d\zeta$ where a and b are independent of τ and $d\zeta$ is a Gauss-Wiener process. Using Ito's Lemma the

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-17- for example, we know that $\nu \to 0$ as $\tau \to 0$ since the return over the last instant before the bond matures must be r for certain.

dynamics for P can be determined

$$(28) \qquad \frac{dP}{P} = \left(R - a\tau + \tfrac{1}{2} b^2 \tau^2 \right) dt - b\tau \, d\xi$$

$$and \qquad \nu^2 = \tau^2 \, Var\left[b(R) \, d\xi \right] = \tau^2 \, Var\left[dR \right]$$

$$\rho\nu\sigma = -\tau \, Cov\left[b(R) \, d\xi, \sigma d z \right] = -\tau \, Cov\left(dR, \frac{dv}{v} \right)$$

The variance and covariance terms do not depend on
maturity; therefore, they can be estimated from a time
series of long interest rates.-18- The time series of
changes in the interest rate demonstrated heteroscedasticity
that was approximately linear (ie. b(R) = kR); therefore,
the estimates of ν and ρ employed were

$$\hat{\nu}^2 = R^2 \tau^2 \, \hat{Var}\left(\frac{dR}{R} \right) \equiv R^2 \tau^2 \hat{\Sigma}^2$$

$$(29)$$

$$\hat{\rho} = -\frac{1}{\hat{\sigma}\hat{\nu}} R\tau \, \hat{Cov}\left(\frac{dR}{R}, \frac{dv}{v} \right) \equiv -R\tau \, \hat{\Gamma}/\hat{\sigma}\hat{\nu}$$

Performing the integration for U*

$$(30) \qquad U^* = \hat{\sigma}^2 \tau + \tfrac{1}{3} R^2 \tau^3 \hat{\Sigma}^2 + R\tau^2 \hat{\Gamma}$$

The sample variance over the seven year period for dR/R was
2.58×10^{-4}. The sample covariances ranged from -3.70×10^{-5}
for Leverage to -9.48×10^{-5} for Hemisphere. Since $R\tau < 1$
throughout the sample period, the addition to U* due to
the stochastic nature of the interest rate was very small.

-18- It is here that the assumption that b(R) is
independent of τ is required.

The largest difference in model values was less than ten
cents.

The stochastic nature of the interest rate is not the
only misspecification in the model which could result in a
error of the type described that would apply uniformly to
all the dual funds. A personal income tax can be shown to
have very similar effects.

In a world in which a personal income tax is assessed,
it is reasonable to assume that investors are concerned with
their after-tax return on investment and that a contingent
claim should therefore be priced by its relative after-tax
value. If the returns on two claims are subject to
different taxes, then their relative prices will depend upon
the taxes assessed.

As a simple model assume that ordinary income is taxed
at the rate T, the same for all income levels and constant
over time. Capital gains are not taxed at all. If an
investor forms a portfolio with W_1 dollars in the dual
fund's assets, W_2 dollars in the contingent claim (ie. the
income or the capital shares), and W_3 dollars in a
riskless bond paying taxable interest continuously at the
rate r, then his after-tax return will be:

$$(31) \qquad dx = \frac{dV + C(1-T)dt}{V} w_1 + \frac{dY + c(1-T)dt}{Y} w_2 + r(1-T)w_3 \, dt$$

This equation is identical to equation (4) in $|5|$ where C,
c, and r -19- are replaced by their after-tax values, hence

-19- We assume that the returns used to pay the
management fees are taxable to an investor holding the dual

we can immediately write the implied differential equation
in P = Y.

$$(32) \qquad \frac{1}{2}\sigma^2 V^2 F_{VV} + (1-T)(r-\delta)V F_V - r(1-T) F - F_\tau + (1-T) c = 0$$

Under the same assumptions as before, the value of the
capital shares is

$$(33) \qquad f(V,\tau) = w\left(V e^{-\delta(1-T)\tau}, \tau ; E, (1-T)r, \sigma^2\right)$$

The marginal impact of the tax is

$$(34) \qquad f_T = \tau f q \left[\delta - r\left(1 - \frac{1}{q}\right)\right]$$

Ceteris paribus the impact will be larger in absolute value
earlier in the fund's history; however, the sign of the
effect is uncertain. $f_T \gtrless 0$ as $\delta \gtrless r(1-\frac{1}{q})$. For the no
dividend case the lower inequalities hold. If $\delta \geq r$,
then the upper set will hold. Both cases follow upon
observing that q is in the interval $(1,\infty)$. From section
IV:

$$(35)$$

$$\lim_{z \to 0} q = \infty \qquad\qquad f_T \sim \tau(\delta-r) V e^{-\delta\tau}\phi(x_1) \gtrless 0$$

$$\lim_{z \to \infty} q = 1 \qquad\qquad f_T \sim \tau V e^{-\delta\tau} > 0$$

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fund assets. This is the general case for dual funds since
the management fees are paid out of the investment income.
For Gemini and Hemisphere the management fees are paid
equally out of capital gains and income. In this case the
first term in (31) becomes $(dV + C(1-T/2) - DT/2)/V$.
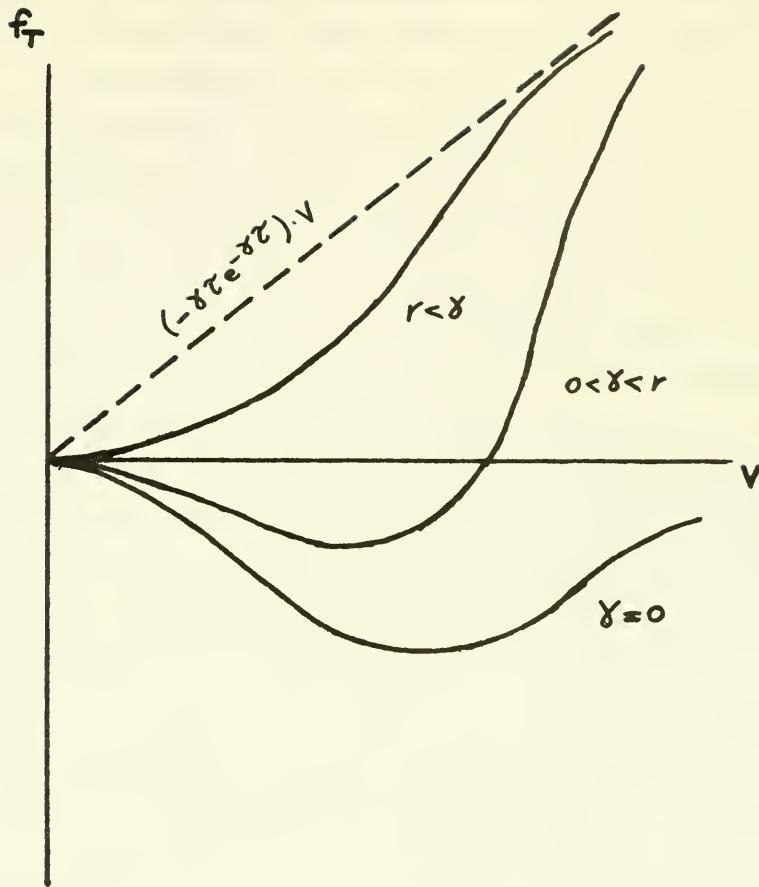
Figure 13

hence in the intermediate case, $0 < \gamma < r$, f_T will take on first negative and then positive values as V increases. As we saw in sections IV and V this last is the relevant case.

The uncertainty in the sign of f_T may seem counterintuitive. One might expect the capital shares which escape taxation to benefit at the expense of the income shares. This effect is present; nevertheless, it may be smaller than the relative decrease in value due to the smaller effective (ie. after-tax) interest rate.-20- When dividends are large (ie. V is large) then the former effect will dominate and vice versa.

Ignoring for the moment the non-dividend cash payments of the dual funds, which are minor, we can see immediately that the capital shares must be an increasing function of the tax rate whenever the current yield on the income shares is greater than the riskless rate. Ie. if $\gamma V/(V-f) > r$, then $\gamma > r(1-f/V) > r(1-f/Vf_V) = r(1-1/q)$. In principle current yields on the income shares need not exceed r. For example, if the income shares were selling at a discount and a capital gain were expected or if the growth in dividends expected through the growth in asset value were sufficiently large, the current yield might be smaller than the interest rate. However, the dividend yields did exceed the riskless rate in general for the sample period. Hence introducing taxes will increase the capital share prices and do so to a greater extent for the longer maturities.

Table 6 shows statistics for this model. A comparison

-20- The capital share price is an increasing function of the interest rate. See (8d).

Table 6

Comparison of Models Incorporating Personal Income Tax

	A	G	H	I	L	P	S
				Tax rate = 50%			
R^2	0.985	0.767	0.985	0.963	0.904	0.971	0.851
RMSE	0.396	1.096	0.656	0.864	1.406	0.490	0.581
Mean Error	1.339	0.238	0.536	0.558	1.324	0.356	0.228
Estimated slope	1.027	0.996	0.977	1.192	0.943	0.964	0.701
Fraction of Error due to:							
Bias	0.920	0.047	0.666	0.417	0.886	0.528	0.153
$b \neq 1$	0.004	0.000	0.012	0.234	0.004	0.021	0.430
Residual variance	0.076	0.953	0.322	0.350	0.110	0.451	0.417
Misspec. Error	0.145	0.056	0.291	0.485	1.759	0.131	0.196
Residual Error	0.011	1.144	0.138	0.261	0.217	0.108	0.140

	A	G	H	I	L	P	S
				Tax rate = 25%			
R^2	0.983	0.737	0.988	0.948	0.911	0.970	0.839
RMSE	1.793	1.796	1.021	1.447	1.947	0.735	0.692
Mean Error	1.722	1.355	0.930	1.179	1.890	0.648	0.499
Estimated slope	1.104	1.035	1.095	1.294	1.098	1.060	0.760
Fraction of error due to:							
Bias	0.923	0.587	0.828	0.663	0.942	0.778	0.520
$b \neq 1$	0.026	0.001	0.066	0.163	0.004	0.021	0.164
Residual variance	0.051	0.412	0.105	0.174	0.053	0.201	0.316
Misspec. Error	3.050	1.896	0.931	1.729	3.586	0.431	0.327
Residual Error	0.163	1.328	0.109	0.364	0.200	0.108	0.151

Note: For an explanation of the items in this Table

please see Appendix C.

with table 4 shows the expected decrease in mean square
error as we consider the tax rates 0%, 25%, and 50%. Even
in this last case, however, each model was on average too
low. Although tax rates greater than 50% are possible in
the U.S. and such higher rates would reduce the model-market
discrepancy even more, 50% was the upper limit considered
here for $\land$ reasons. (1) Those investors in the very high
tax brackets generally have their investments personally
managed rather than using mutual funds as investment
vehicles. (2) Excluding the capital gains tax in the model
formulation as we have done tends to overstate the impact of
a tax.-21- (3) The income shares of these funds were largely
bought by corporations which are allowed an 85% exclusion of
dividends paid to them on their corporate taxes.
Consequently the effective tax rate on dividens is only 7.5%
for corporations with a corporate tax rate of 50%.

Inclusion of a tax in the model has helped to reduce
the discrepancy in the manner sought; nevertheless, it
appears that only a partial correction is possible under the
best of circumstances.

-21- If a capital gains tax is also assessed and
collected continuously then (1-T) in the model is everywhere
replaced by $(1-T_i)/(1-T_c)$ where T_i is the tax rate on
ordinary income and T_c is the tax rate on capital gains.
For a capital gains tax of Max(.25,T_i/2) our computed
models then give results for investors in the 40% and 72.5%
tax brackets. While 25% and 50% are underestimates of the
real tax brackets modeled, 40% and 72.5% are overestimates
since capital gains are payable only when the gain is
realized.

VIII Model Error--Misspecification

The alterations to the basic dual fund model examined in the previous section were all accomplished through parameter change. In this section we turn our attention to model error caused by misspecification. In particular, assumption 3 about the asset return dynamics will be examined.

The derived pricing equation (7) is based on lognormal return dynamics with a constant variance. In section V evidence was presented suggesting that this simple assumption is not strictly true, and two explanations were offered. To review, the first postulated that dz is drawn from a stable distribution with an infinite second moment. The second argues that dz is as postulated in A3; however, σ^2 is not a constant. The first hypothesis is probably more damaging to the model presented here. In this case Ito's Lemma may not be used to deduce the contingent claims' price dynamics. In addition the expected change in asset value over any interval would be infinite as noted by Samuelson [9; footnote 1]. Under such conditions, Merton [4; footnote 42] has conjectured that the only equilibrium option price would be the stock price itself. In the model here this would imply that f = V and F = 0. Under the second explantion the exact nature of the process driving σ^2 is important. If the variance is a known function of time, then the entire derivation is valid and $\sigma^2\tau$ in (7) is replaced by a generalized uncertainty term $U \equiv \int_0^\tau \sigma^2(s)ds$ much as in the

case of a stochastic interest rate. If the variance is stochastic but it is a known function of V (and possibly time), then the derivation of (2) is valid; however, it does not have the simple closed form solution (7). Finally if the variance is stochastic even conditionally on V then the hedging derivation breaks down entirely as no portfolio will be completely riskless. Even in this last case if σ^2 is not "too stochastic" in nature, then the model may closely approximate the true solution.

Determining which explanation is more plausible is important if we are to decide whether this model can be improved upon or should be abandoned. If the log price changes are drawn from a stable distribution other than the normal (ie. the characteristic exponent α is less than 2), then the population variance is infinite. However, as long as it is greater than 1, scale parameters of degree one (eg. mean absolute deviation, interquartile range, etc.) do have finite expectations. If S_n is such a scale paramter based upon independent changes over n weeks, then Mandelbrot |3| | has shown that

$$(36) \qquad S_n = n^{1/\alpha} S_1$$

From this result we can deduce that $S_1\sqrt{n}/S_n$ should be a decreasing function of n reaching zero in the limit for all stable distributions except the normal.

Table 7 presents this ratio based on the mean absolute deviation for n=1 to 20 weeks. This evidence offers little support to the hypothesis that the distribution of

TABLE 7

RESULTS OF SCALE PARAMETER TEST

$$S_1 \sqrt{n}/S_n$$

N	A	O	H	I	L	S
1.	1.000	1.000	1.000	1.000	1.000	1.000
2.	1.019	0.958	1.047	1.101	1.120	0.989
3.	0.903	0.949	1.131	0.984	0.987	1.019
4.	1.135	1.053	1.014	0.972	1.051	0.957
5.	0.999	0.961	1.181	0.908	1.052	1.044
6.	1.058	0.863	1.093	0.953	1.079	1.032
7.	0.879	1.004	1.210	1.013	1.127	1.084
8.	1.060	1.040	1.123	1.040	1.222	1.048
9.	0.989	0.907	0.999	0.931	1.012	0.892
10.	1.109	1.125	1.323	1.203	1.214	1.064
11.	1.021	0.958	0.998	0.895	1.107	1.108
12.	0.942	0.870	0.946	0.947	1.065	0.903
13.	0.933	1.171	1.196	1.241	1.063	1.138
14.	1.092	1.123	1.157	1.234	1.160	1.314
15.	0.979	1.130	1.200	0.958	0.971	1.135
16.	0.984	1.016	0.976	0.988	1.015	1.091
17.	1.000	1.190	1.518	1.104	1.225	1.866
18.	0.958	0.995	1.176	1.019	1.037	0.940
19.	1.041	1.131	1.215	1.191	1.223	1.600
20.	0.885	1.060	1.132	1.011	1.021	0.929

log price changes is a non-normal member of the Pareto-Levy family.

The alternate hypothesis is somewhat more difficult to test with no further knowledge of the process controlling the variance. Rosenberg |6| has postulated that this process is governed by general market forces. If this is the case, then we might expect that the computed sample variances for the several funds would tend to move together. To test this possibility the sample variances were computed every year for each fund and a two-way analysis of variance was performed. The F-statistic across years was significant at the one percent level indicating that some market factors probably were affecting all the funds in a similar manner. It is not clear if the effect was directly caused by the co-movement of the underlying assets' variances or if it was due to similar reactions of the funds' managers to the market. This distinction, however, is immaterial to the pricing of the dual funds ex post.

Although the evidence presented here is more supportive of the Rosenberg hypothesis than of Mandelbrot's, we have no clue as to how to proceed to improve upon the model already presented so we turn our attention to another matter.

The dividend policy assumed in the model was a proportional one. While this appears to be a reasonable assumption, it was chosen primarily because an analytic solution is known for equation (2) only in this case.-22- A

-22- An additional assumption not explicitly stated in A3 is that the dividend policy is not stochastic. Ie. the dividend payment is known exactly given the asset value. If this were not the case, then the share prices would also be

more realistic assumption would be that dividends are linear
in asset value $(C=a+bV \quad a,b > 0)$ In this case (2) becomes:

$$(37) \qquad \tfrac{1}{2} \sigma^2 V^2 \bar{f}_{vv} + (rV - a - bV)\bar{f}_v - r\bar{f} - \tilde{f}_\tau = 0$$

while an analytic solution to (37) is not known, we can
compute an asymptotic value for $\bar{f}$ for large V. First
transform the equation by making the substitutions
$Y=V\exp(-b\tau)$ and $\psi(Y,\tau)=\bar{f}$. Then (37) becomes

$$(38) \qquad \tfrac{1}{2} \sigma^2 Y^2 \psi_{YY} + (rY - a e^{-b\tau})\psi_Y - r\psi - \psi_\tau = 0$$

We can now consider ψ as the capital share on a fund with
asset value Y and paying dividends, $C=a \cdot \exp(-b\tau)$ which
are a function of maturity only. Consequently as $Y \to \infty$,
the income share Ψ will approach its limiting value equal
to the future dividends and redemption value discounted at
the riskless rate.

$$(39) \qquad \lim_{Y \to \infty} \Psi = E e^{-r\tau} + \frac{a}{r-b} e^{-r\tau}\left[e^{(r-b)\tau} - 1\right] \equiv E e^{-r\tau} + L(\tau)$$

The asymptotic behavior of ψ is then $\psi \sim Y - \Psi$ and

$$(40) \qquad \bar{f} \sim V e^{-b\tau} - E e^{-r\tau} - L(\tau)$$

If we approximate $\bar{f}$ by f, the proprotional dividend
solution where $Y = C/V = b + a/V$, then expanding f

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dependent upon the stochastic process generating the
dividend stream.

$$(41) \qquad f \sim Ve^{-b\tau}\left[1 - \frac{a\tau}{V} + \cdots\right] - Ee^{-r\tau}$$

and

$$(42) \qquad f - \bar{f} \sim \frac{ae^{-r\tau}}{r-b}\left[\frac{1-(r-b)\tau}{\exp[(b-r)\tau]} - 1\right] \gtrless 0 \quad \text{as } r \lessgtr b$$

If f is an improved model of the capital shares, then (42) is consistent with the observed errors since r>γ>b. It is, however, only an asymptotic result. In general (37) must be solved by numerical integration. This technique can handle problems of a quite arbitrary form; however, a complete solution (ie. for all V and τ) must always be computed. Single values of f(V,τ) can not be obtained.

To isolate the effect of the linear dividends from those of the other factors still not explained by the simple model, the following proceedure was employed. The numerical solution was computed for a fictitious dual fund which completely satisfied the assumptions A1 - A5 and had dividends of the form C=a+bV.-23- Each value of $\bar{f}$ was then compared to that obtained from the approximation f(V,τ) with γ=C/V=b+a/V.

Table 8 presents a representative sample of values for f and $\bar{f}$. The latter is uniformly greater in value than the former (except at τ =0), and the difference increase with maturity for any asset value. Since this is the same

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-23- The values chosen for the parameters were: σ^2 =.035, r=.08, a=.50, b=.01, and E=10. The first four were chosen as representative of the values actually found. The last is merely a scaling factor.

TABLE 3

CAPITAL SHARE PRICES COMPUTED FROM
LINEAR DIVIDEND MODEL (UPPER)
AND PROPORTIONAL APPROXIMATION

TAU=	5	10	15	20	V = 25	30	35	40	45	50
0	0.00 / 0.00	3.00 / 3.00	5.00 / 5.00	10.00 / 10.00	15.00 / 15.00	20.00 / 20.00	25.00 / 25.00	30.00 / 30.00	35.00 / 35.00	40.00 / 40.00
2	0.00 / 0.00	1.16 / 1.11	5.31 / 5.27	10.17 / 10.13	15.07 / 15.02	19.97 / 19.92	24.86 / 24.82	29.76 / 29.72	34.66 / 34.62	39.55 / 39.52
4	0.03 / 0.01	1.02 / 1.46	5.62 / 5.46	10.30 / 10.14	15.05 / 14.71	19.80 / 19.70	24.67 / 24.50	29.47 / 29.30	34.28 / 34.09	39.08 / 38.89
6	0.07 / 0.02	1.71 / 1.04	5.04 / 5.53	10.39 / 10.05	15.06 / 14.70	19.75 / 19.38	24.45 / 24.07	29.16 / 28.76	33.87 / 33.46	38.58 / 38.16
8	0.10 / 0.03	2.12 / 1.71	0.00 / 5.50	10.43 / 9.58	14.77 / 14.40	19.57 / 18.97	24.26 / 23.55	28.81 / 28.14	33.43 / 32.73	38.06 / 37.33
10	0.13 / 0.03	2.27 / 1.72	0.10 / 5.40	10.44 / 9.05	14.76 / 14.04	19.40 / 18.47	23.92 / 22.96	28.44 / 27.45	32.97 / 31.94	37.51 / 36.44
12	0.17 / 0.04	2.38 / 1.70	0.16 / 5.26	10.41 / 9.36	14.75 / 13.63	19.17 / 17.90	23.62 / 22.33	28.05 / 26.71	32.49 / 31.10	36.93 / 35.50
14	0.17 / 0.04	2.45 / 1.65	0.19 / 5.07	10.35 / 9.04	14.74 / 13.18	18.96 / 17.40	23.30 / 21.65	27.04 / 25.93	31.99 / 30.22	36.35 / 34.53

pattern as found for the model-market discrepancy, there is reason to believe that the linear dividend soultion for each fund would show improved predictive ability.

With this encouragement, a numerical solution for each fund could be computed every week. There are however two drawbacks to this idea. First the dividend parameters a and b can be computed only with the knowledge of the entire time series of values and dividends, and even then, only seven data points exist for the linear regression due to the manner in which the dividends are paid. Under this circumstance the linear dividend model might have an unfair advantage in a comparison with the original model which uses only past data to estimate its parameters. On the other hand, the proportionality estimation scheme imposes a structure which must produce a reasonable "expected" dividend policy while the linear estimation does not. In particular, the estimated policy for Putnam and Scudder is certainly not a good estimation of the investors' ex ante expected dividend policy. Second the original model was computed using values for the interest rate and the variance which were updated weekly. To follow a similar proceedure for the new model would present a formidable task since, as mentioned before, a complete solution for f would have to be computed each week requiring some 2400 numerical integrations be performed.

Nevertheless, it is desireable to have a comparison between the linear dividend model and the market price that is more revealing than simply stating that the discrepancy between each one and the original model seems to behave in

the same manner. Consequently the following compromise scheme was employed. Each value from the original model was corrected by a multiplicative factor equal to the quotient of the linear-dividend and proportional-dividend models computed from the fictitious funds at the same asset value-24- and maturity.

Although this method is heuristic, it should give us a rough evaluation of the linear dividend model since the interest rate and the variance estimates did not change greatly over time. The fictitious a and b are close to the estimated values for American, Hemisphere, Income & Capital, and Leverage. More importantly, they should be close to the ex ante expected values for all the funds.

Figures 14 to 20 show the new model as well as the proportional model and market price for the capital shares of each dual fund. The linear dividend model seems to be better as Table 9 confirms.

-24- The asset values were first normalized on E=10 to scale then to the fictitious fund.

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Table 9

Statistics for Linear Dividend Model

	A	G	H	I	L	P	S
R^2	0.969	0.810	0.965	0.963	0.910	0.950	0.716
RMSE	1.346	1.396	0.765	1.162	1.382	0.434	0.704
Mean Error	1.209	0.940	-0.377	0.818	1.275	-0.046	-0.113
Estimated Slope	1.078	1.226	1.133	1.334	1.244	1.008	0.639
Fraction of Error due to:							
Bias	0.808	0.453	0.244	0.495	0.852	0.011	0.026
b≠1	0.027	0.069	0.209	0.312	0.041	0.001	0.435
Residual variance	0.165	0.477	0.547	0.193	0.107	0.987	0.539
Misspec. Error	1.511	1.018	0.262	1.087	1.705	0.002	0.227
Residual Error	0.298	0.929	0.320	0.261	0.203	0.185	0.267

Note: For an explanation of the items in this table
please see Appendix C.

FIGURE 14

AMERICAN DUALVEST

- Capital Share Market Price

o Market Based Model

x Corrected Model

FIGURE 15

GEMINI FUND

- Capital Share Market Price

o Market Based Model

x Corrected Model
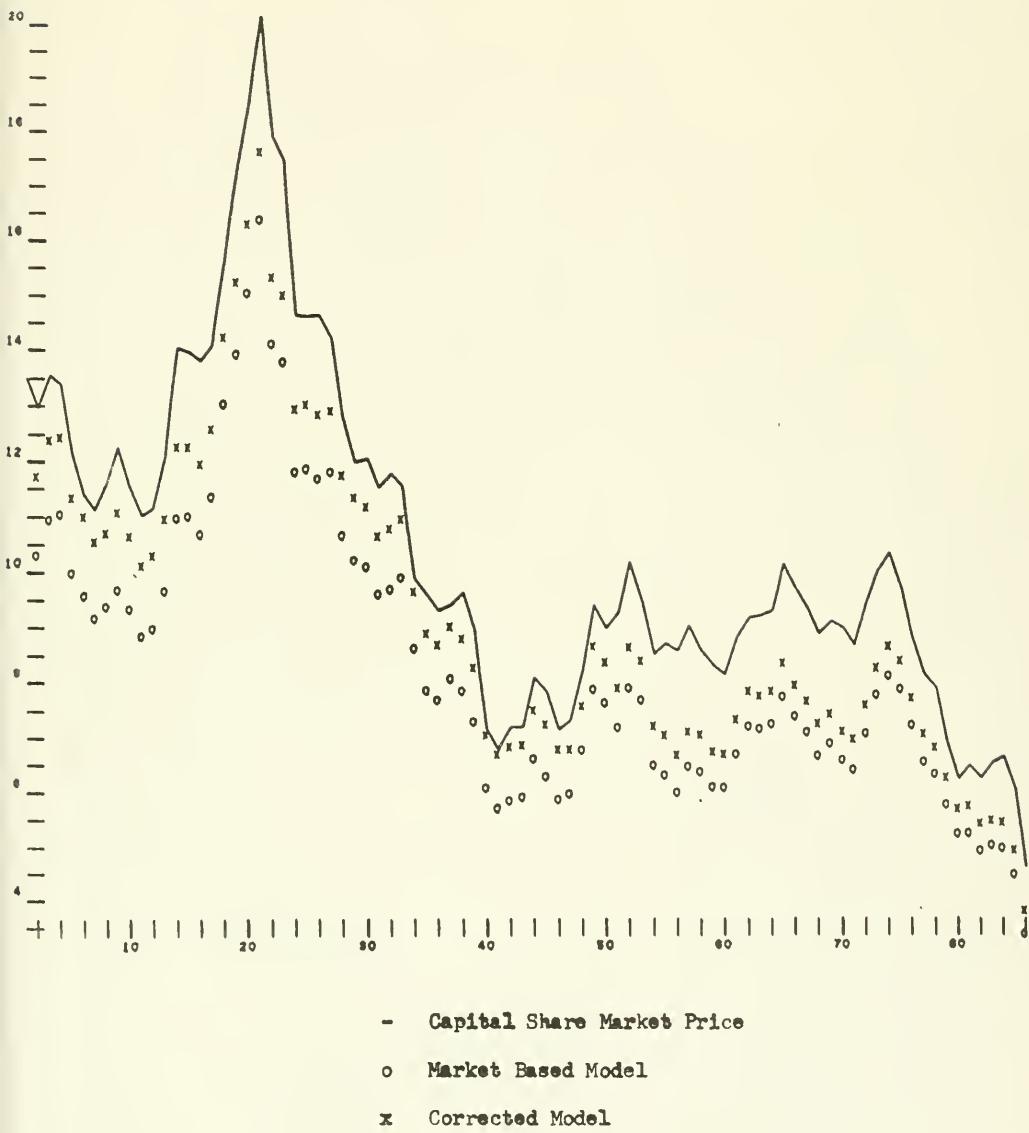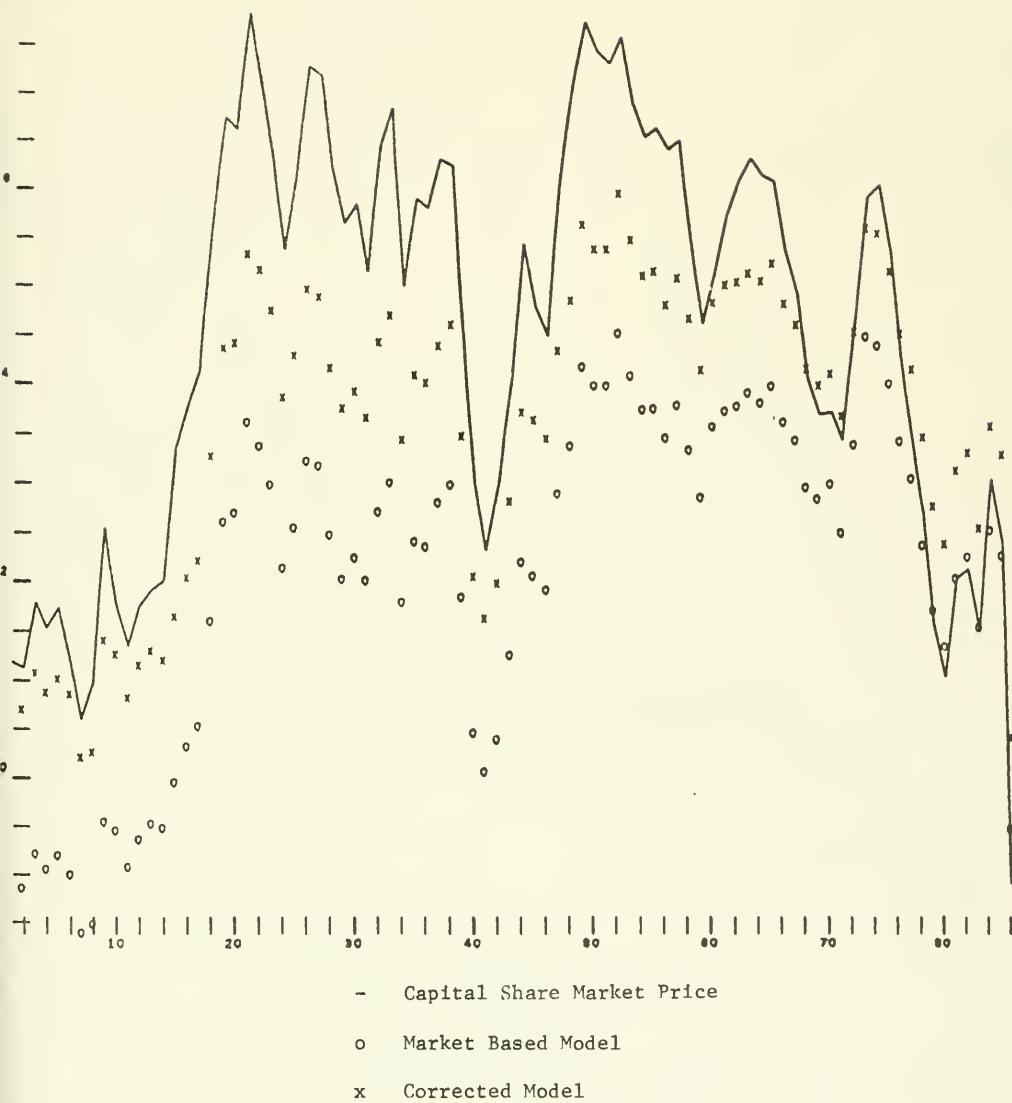
FIGURE 16

HEMISPHERE FUND

-   Capital Share Market Price
o   Market Based Model
✗   Corrected Model

FIGURE 17

INCOME & CAPITAL SHARES

-     Capital Share Market Price

o   Market Based Model

x   Corrected Model

FIGURE 18

VERAGE FUND

- Capital Share Market Price

o Market Based Model

x Corrected Model
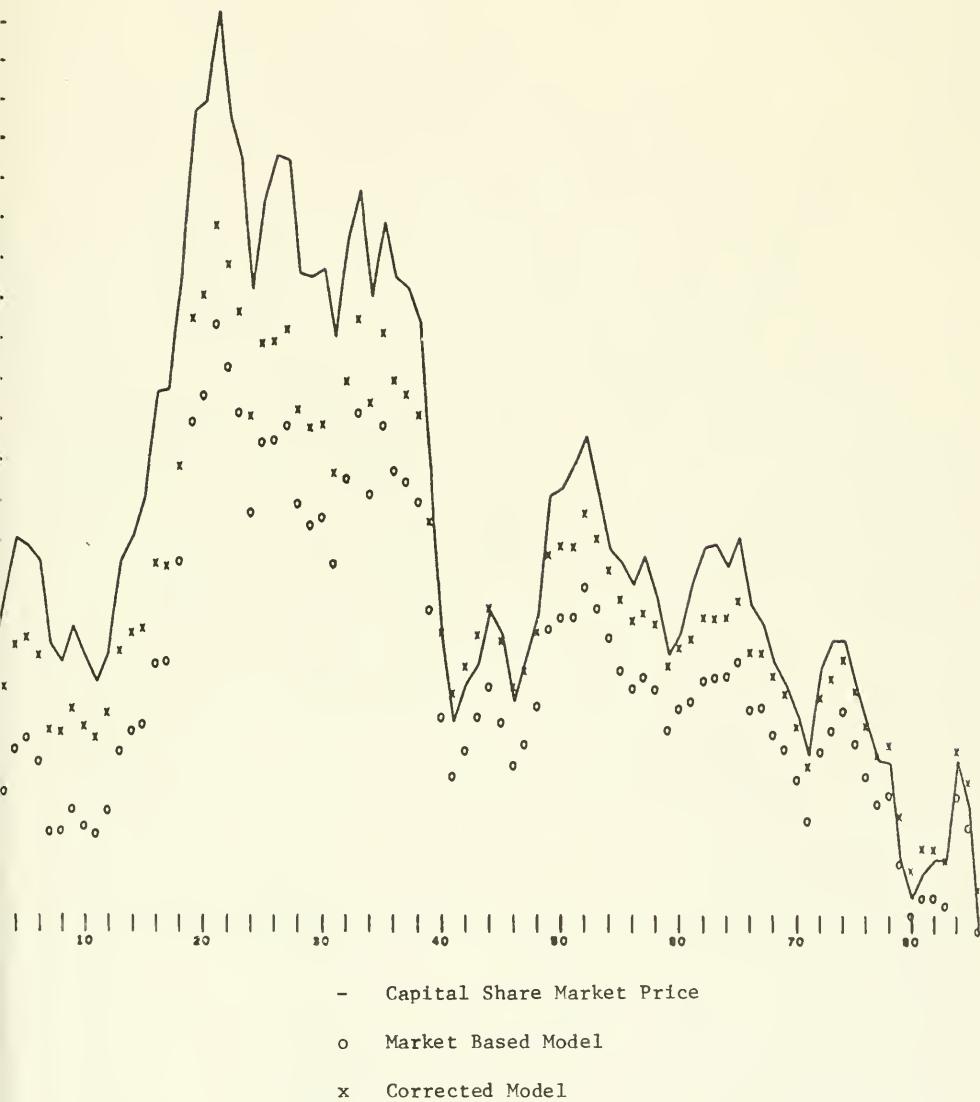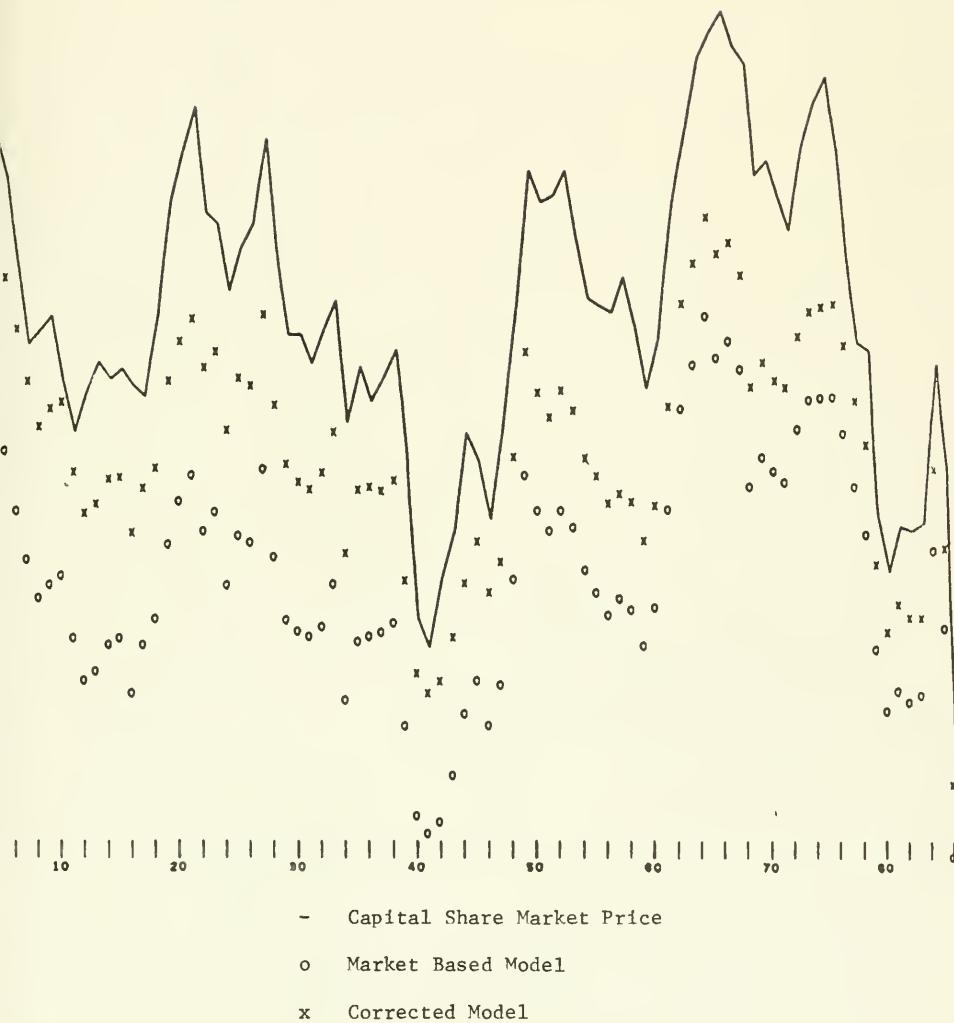
FIGURE 19

PUTNAM DUOFUND

-    Capital Share Market Price

0   Market Based Model

x   Corrected Model

FIGURE 20

SCUDDER DOU-VEST

- Capital Share Market Price

o Market Based Model

x Corrected Model
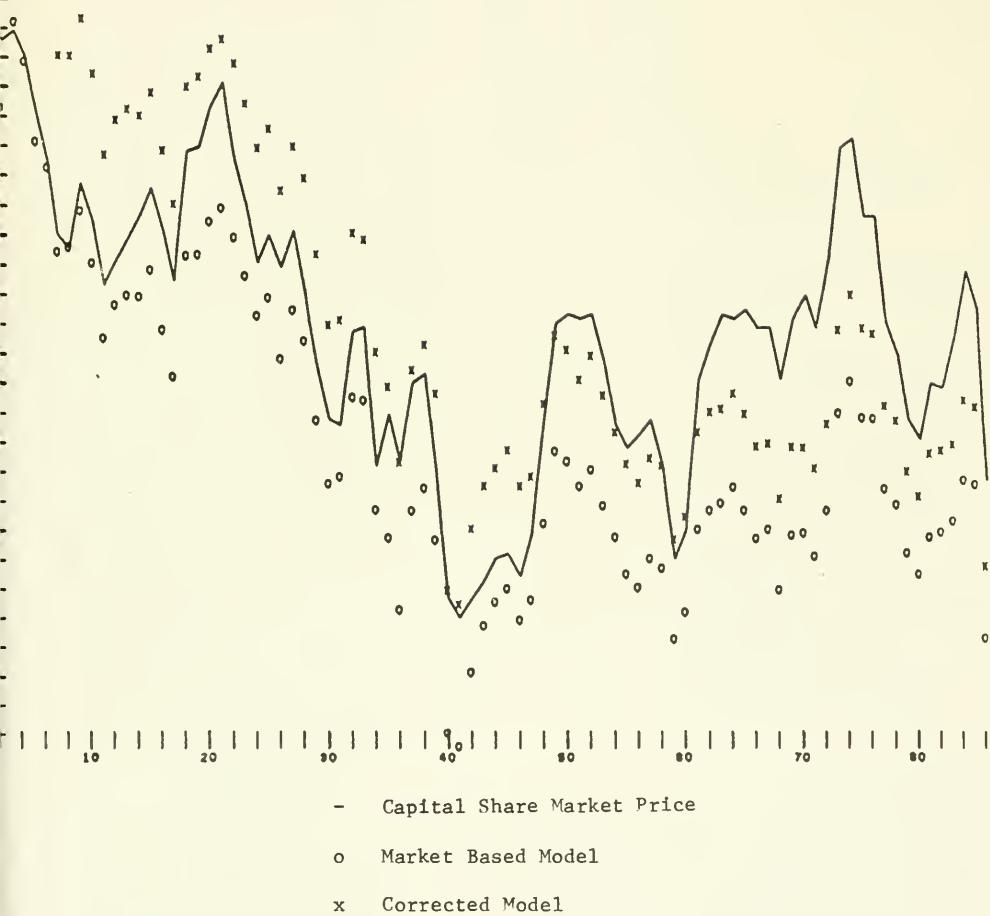
IX Conclusion

The first four secions of this paper formulated a dual purpose fund pricing function based on the option studies of Black & Scholes and Merton. The characteristics of this function were examined in order to explain the behavior of the share prices of the dual funds' claims. In particular it was found that under the idealized conditions within the model any closed-end fund should sell at a discount from its asset value. It is also not inconsistent to find the capital shares selling at a price below their net asset value. This latter fact, in particular, has not heretofore been fully appreciated.

The formualted model was tested in the later sections. There it was found that the model predicted price fluctuations in the capital shares quite well. There was, however, evidence of misspecification in the model. Several methods of eliminating the misspecification were examined. The two most promising alterations were the inclusion of a simplified income tax in the model and a more realistic dividend policy. We can conclude that analysis of this type can be quite useful in the pricing of contingent claims other than the simple option contracts previously examined by Black and Scholes.

Further work is required to ascertain that these results were not influenced by the bias inherent in the ex-post specification process. (Ie. in the correction for observed errors rather than a better ex-ante specification.) Also study into the improved incorporation of asset value,

rather than market value, in the pricing function would be desireable as this could be applied to all closed end investment companies to explain the encountered discounts and premiums.

Appendix A

This appendix presents a formal derivation of the contingent claims pricing equation used in the text along the lines of the option pricing method in |4|. The assumptions stated in section II are assumed throughout with the substitution of

A 4   The price of a riskless in terms of default discount bond promising one dollar τ periods from now is $P(\tau)$. The dyanmics of the returns on this bond are described by

$$\frac{dP}{P} = \pi dt + \nu d\varsigma$$

where π and ν^2 are the instantaneous expected return and variance of return on the bond and $d\varsigma$ is a Gauss-Wiener process.

If $F(V, P, \tau)$ is the contingent claim to be priced, then by Ito's Lemma

(A1)
$$dF = F_V \, dV + F_P \, dP - F_\tau dt + \tfrac{1}{2}\left[F_{VV} \, dV^2 + 2 F_{VP} \, dV dP + F_{PP} \, dP^2 \right]$$
$$= \beta F dt + \lambda F \, dz + \zeta F d\varsigma$$

where

$$\beta \equiv \tfrac{1}{2}\sigma^2 V^2 F_{VV} + g\sigma\nu PV F_{VP} + \tfrac{1}{2}\nu^2 P^2 F_{PP} + (dV - c) F_V + \pi P F_P - F_\tau$$

$$\lambda \equiv \sigma V F_V / F$$

$$\zeta \equiv \nu P F_P / F$$

$$g \equiv Cov(dz, d\varsigma)$$

We now form a portfolio with W_1 dollars in vested in the asset, W_2 dcllars invested in the contingent claim, and W_3 dollars invested in the bond. The dollar return on this portfolio will be

(A2)
$$dx = w_1 \, \frac{dV + C dt}{V} + w_2 \, \frac{dF + c dt}{F} + w_3 \frac{dP}{P}$$

Constrainig the portfolio to have zero net investment (ie. W_1 + W_2 + W_3 = 0) and substituting for dV, dP, and dP we have

$$(A3) \quad dx = \left[w_1(\alpha - \pi) + w_2(\beta - \pi + c/F) \right] dt + \left[w_1 \sigma + w_2 \lambda \right] dz - \left[w_1 \nu + w_2(\nu - 3) \right] d\varsigma$$

If we now choose W_1 and W_2 to eliminate the uncertainty in dx, then the portfolio will be riskless and since it requires no investment, its expected return must be zero. These conditions can be expressed as the system of equations.

$$(A4) \quad \begin{array}{lll} (\alpha - \pi)\, w_1 & + & (\beta - \pi + c/F)\, w_2 = 0 \\ \sigma\, w_1 & + & \lambda\, w_2 = 0 \\ \nu\, w_1 & + & (\nu - 3)\, w_2 = 0 \end{array}$$

A non-trivial solution set to (A4) wil exist only if

$$(A5) \quad \frac{\beta - \pi + c/F}{\alpha - \pi} = \frac{\lambda}{\sigma} = \frac{\nu - 3}{\nu}$$

The first condition in (A5) together with the definitions of β and λ implies that

$$(A6) \quad \tfrac{1}{2}\sigma^2 V^2 F_{VV} + \beta\sigma\nu\,\hat{P}VF_{PF} + \tfrac{1}{2}\nu^2 P^2 F_{PP} - C F_V - F_\tau + c - \pi(F - VF_V - PF_P) = 0$$

If the bond price dynamics are not stochastic then $F_P = F_{PP} = F_{VP} = 0$ If the term structure is flat then $\pi = r$. In this case (A6) reduces to

$$(A6') \quad \tfrac{1}{2}\sigma^2 V^2 F_{VV} + (rV - c) F_V - rF - F_\tau + c = 0$$

which is equation (2) in the text. In the general case (A6) can be simplified since the term in parenthesis is zero from

the second condition in (A5) and the defintions of ____ and

(A7) $\qquad \dfrac{V F_V}{F} = \dfrac{\lambda}{\sigma} = \dfrac{\nu - \xi}{\nu} = \dfrac{F - P F_P}{F}$

Therefore,

(A8) $\qquad \frac{1}{2} \sigma^2 V^2 F_{VV} + \rho \sigma \nu V P F_{VP} + \frac{1}{2} \nu^2 P^2 F_{PP} - C F_V - F_\tau + c = 0$

For a capital share $c = 0$ and making the substituions $X = e^{-\delta \tau} V$ and $G(X, P, \tau) = F(V, P, \tau)$, we derive

(A9) $\qquad \frac{1}{2} \sigma^2 X^2 G_{XX} + \rho \sigma \nu X P G_{XP} + \frac{1}{2} \nu^2 P^2 G_{PP} - G_\tau = 0$

which is identical to equation (34) in |4|. Therefore, we can immediately write its solution

$$ f^*(V, P, \tau) = E P(\tau) W(Z^*, U^*; 1, 0, 1) $$

where

$$ Z^* \equiv \dfrac{V e^{-\delta \tau}}{E P(\tau)} $$

$$ U^* \equiv \int_0^\tau \left[\sigma^2(s) - 2 \rho(s) \nu(s) \sigma(s) + \nu^2(s) \right] ds $$

Appendix B

In this appendix we shall demonstrate that even the capital shares of a dual fund holding only riskless assets and liable for no management fees may still be subject to a discount.

The dividend policy of a dualfund can be expressed in general as $C=C(V,t)+e$. If the dividend policy is certain conditionally on V and t (ie. the error is zero), then in the case of a riskless fund, the explicit value dependence may be supressed since V itself is a deterministic function of time. If the dividend policy $C(t)$ is an equilibrium policy that allows the income share offering price to be equal to E, the redemption price, then the capital shares will sell at a discount at time t if

$$(B1) \qquad \int_0^t c(s) e^{-rs} \, ds \; < \; E(1 - e^{-rt})$$

The left hand side of (B1) is the initial present value (ie. at the offering time) of the dividends that will be paid through time t. The right hand side is the initial present value of a stream of dividends paid continuously at a constant rate of rE per unit time over the same time period. The two sides of equation (B1) must be equal at the offering date of the fund; hence, the capital shares will sell at a discount (premium) if the dividends on the income shares are in arrears (ahead) of the constant rate policy in a present value sense.

The proof of (B1) comes directly from $f=V-F$, $F(0)=E$, $f(0)=E$, and

$$(\text{B2}) \qquad F(t) \; = \; E\,e^{-r(T-t)} \; + \; \int_t^T c(s)\,e^{-r(s-t)}\,ds$$

from which it will be observed that f < V-E whenever (B1)
holds.

If the dividend payments have been small enough in the
past, (B1) is clearly a possible condition. The relevant
question here is under what conditions it will hold for a
fund paying proportional dividends at the equilibrium rate.
Since the fund is riskless $V(t) = 2E\exp((r-\gamma)t)$, $f(t) =$
$E\exp(rt)$, and from (23a) $\gamma = r-\log((1+\exp(rT))/2)/T$ The
capital shares will sell at a discount whenever

$$(\text{B3}) \qquad \left[\frac{1+e^{rT}}{2}\right]^{1/T} \; > \; \left[\frac{1+e^{rt}}{2}\right]^{1/t}$$

but this is always the case since T>t and $\sqrt[t]{(1+\exp(rt))/2}$
is an increasing function of t.-25- We have proved then that
the capital shares of a riskless dual fund paying
proportional dividends at the equilibrium rate will always
sell at a dsicount from net asset value. The behaviour of
the discount over time can be seen in figure B-1. Early in
the fund's history it increases in size reaching a maximum
and then decreasing until it just disappears at the maturity
date.

-25- I thank D. Fehr for a proof of this.
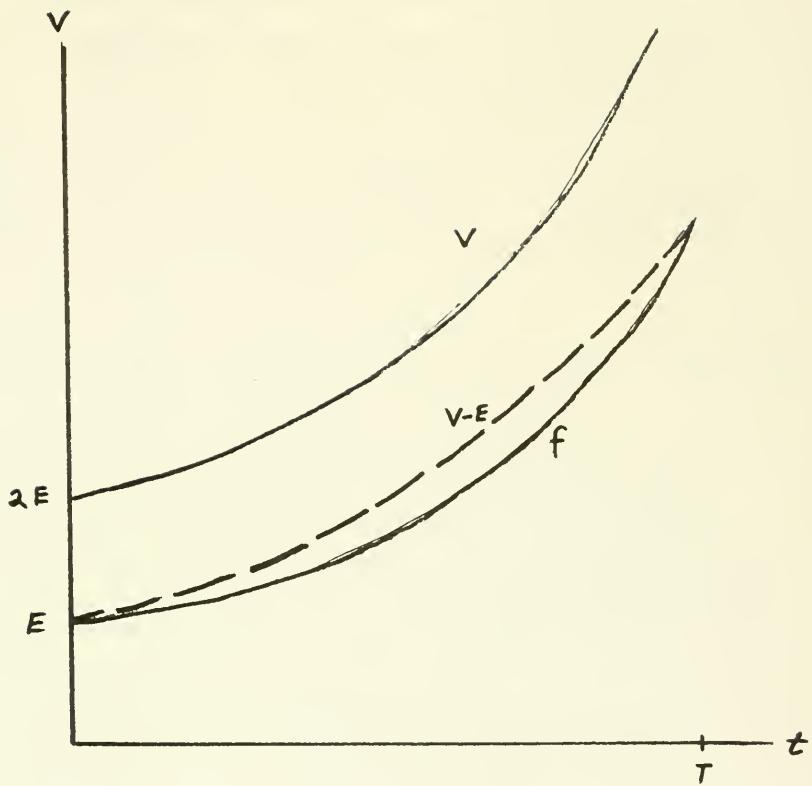
Figure B1

Appendix C

This appendix explains the partioning of the mean
square error between two time series as is found in Tables
3, 6, and 9. If Y and y are an observed and a modeled
time series, and $\hat{y}$ is the time series defined by $\hat{y} = a +$
by where a and b are the coefficients of the regression
of Y on y, then the mean square error may be decomposed
into:

$$(B1) \qquad MSE = \frac{1}{n}\sum(y-Y)^2 = (\bar{y}-\bar{Y})^2 + \frac{1}{n}(1-b)^2\sum(y-\bar{y})^2 + \frac{1}{n}\sum(Y-\hat{y})^2$$

The first term on the right hand side of (B1) is that
portion of the mean square error due to bias. The second
and third terms are those portions due to the difference of
the regression coefficient, b, from unity and due to the
residual variance about the regression line.

Thus the third through sixth rows for each model in
these tables are b, $n(\bar{Y}-\bar{y})^2/MSE$, $(1-b)^2\sum(y-\bar{y})^2/MSE$, and
$\sum(Y-\hat{y})^2/MSE$.

The row labeled "Misspecification Error" is the sum of
the first two terms in (B1). This measures the overall
performance of the model as to correct level (first term)
and response (second term). The final row labeled "Residual
Error" is an indication of the model's ability to predict
individual points in the time series.

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